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THE ACOUSTIC POWER RADIATED BY A CIRCULAR MEMBRANE EXCITED FOR VIBRATION BOTH BY MEANS OF THE EDGE AND BY EXTERNAL SURFACE LOAD

K. SZEMELA, W. P. RDZANEK Jr., W. RDZANEK

University of Rzeszów
Rejtana 16a, 35-310 Rzeszów, Poland
e-mail: alpha@univ.rzeszow.pl

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In this paper the acoustic power of the circular membrane, excited both by the edge and external exciting forces uniformly distributed over the whole surface, is examined. Some different amplitudes of exciting factors and some differences between the phases of excitations were considered. It has been assumed that the source of a sound is located in a flat, rigid and infinite baffle and is sourrounded by a lossless and homogeneous fluid medium. The vibrations are axisymmetric and time-harmonic. Employing the Cauchy's theorem of residues and asymptotic formulae for the Bessel functions, the asymptotes for active and reactive power consisting of elementary functions are obtained. The acoustic power radiated by the membrane was shown graphically in terms of the parameters describing both kinds of excitations.

Key words: acoustic radiation power, vibrations of a spherical membrane, excitation produced by edge and surface loads, amplitude-phase effects.

Notations

a membrane radius,
c propagation velocity of a wave in a fluid medium,
$H_n^{(1)}$ first kind, n-th order Hankel function,
J_n n-th order Bessel function,
k_0 wave number,
$p(\mathbf{r})$ acoustic pressure,
P total sound power radiated by a membrane,
P_{active} active power,
P_{reactive} reactive power,
r radial variable of a point on the surface of the membrane in polar coordinates,
S membrane's area,
s amplitude of the external excitation force per unit area of the surface,
t time,

T streching force per unit length,
η transverse deflection of membrane's points,
ρ_0 rest density of the fluid medium,
σ mass of membrane per unit area,
ω frequency.

1. Introduction

The analysis of magnitudes determining radiation of surface sound sources such as membranes or plates is very important from the practical point of view. The possibility of using purely theoretical results to design an acoustic system, which can exist as damping noise components, is of essential importance. We can influence the radiation of source in adiction to surface excitement, a driving force applied to its edge. In this way the power of radiation is dependent on both the amplitude of excitation at the edge and on its phase.

In the paper [5] the abilities of control amplitudes and phases of both clamped edges of annular plate excited at the same time by external surface force were investigated.

W. J. RDZANEK, W. P. RDZANEK JR, Z. ENGEL have obtained integral formulae, which determine the dependence of the acoustic power radiated by annular plate on the amplitude of excitation of external edge and stiffness constants associated with the boundary conditions [3].

It is very desirable to find elementary formulae describing the influence of parameters connected with a source on the acoustic sound power. This problem is mathematically complicated and it is impossible to obtain some formulations for the power valid for all frequencies.

The analysis of sound power radiated by a circular membrane excited to vibration by the edge and external surface force have not been presented in the literature yet. In spite of its simplicity, this problem has a great practical importance because it can be applied to design and build active damping noise systems.

Making use of Cauchy's theorem, the asymptotes for the sound power were reached in an elementary form. The asymptotes determine the dependence of the acoustic power on parameters characterizing the excitations.

2. Assumptions

The membrane, the radius of which equals a is stretched on the circle by force T referred to unit length. Further it was assumed that its mass per unit area of surface is equal to σ. The source considered is embedded into a flat, rigid and infinite baffle and vibrations are axisymmetric and time-harmonic with frequency ω (Fig. 1). The sound wave is radiated in a homogenous, lossless fluid medium. The external exciting force is uniformly distributed on the whole surface of the membrane and is time-harmonic. This excitation is mathematically described by

$$F_W(t) = se^{-i(\omega t+\varphi)}, \tag{1}$$

where s is amplitude of excitation referred to unit area of the surface, and ω, φ denote frequency and phase.

The second kind of excitation is produced by means of the edge and can be expressed in the form of

$$w(t) = \eta_0 e^{-i(\omega t + \varphi_0)}; \tag{2}$$

by η_0 and φ_0 we have denoted amplitude of excitation and its phase, respectively.

The above equality represents the boundary condition at the edge of membrane.

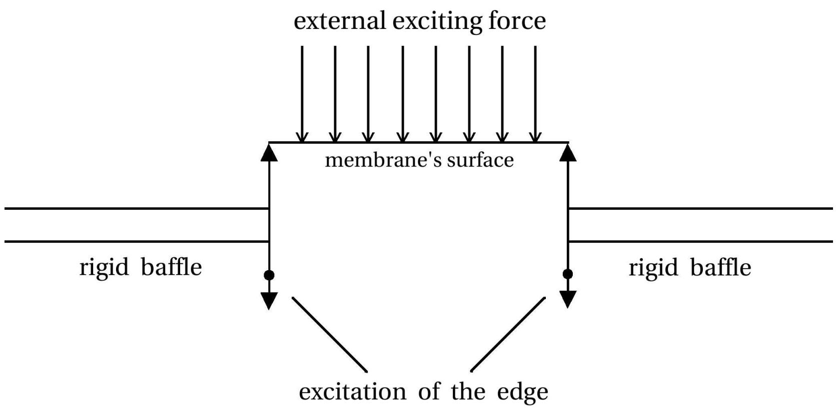

Fig. 1. The configuration of a vibrating system with both kinds of excitations.

For time-harmonic and axisymmetric vibrations a transverse deflection can be formulated as

$$\eta(r,t) = \eta(r) e^{-i\omega t}, \tag{3}$$

where $\eta(r)$ is a function determining the value of a transverse deflection's amplitude in terms of the radial distance.

The equation of motion for the considered source is given by the following formula

$$T\Delta_r \eta(r,t) - \sigma \frac{\partial^2 \eta(r,t)}{\partial t^2} = F_W(t), \tag{4}$$

where Δ_r is the radial component of operator Δ.

On the basis of formulae (1), (2), (3), the equation of motion and boundary condition can be presented in the following form [2, 8]:

$$\left(T\Delta_r + \omega^2 \sigma\right) \eta(r) = s e^{-i\varphi}, \qquad \eta(a) = \eta_0 e^{-i\varphi_0}. \tag{5}$$

Solving the above equation together with the boundary condition and introducing the following notations:

$$f = \frac{s}{Tk^2}, \qquad k^2 = \frac{\omega^2 \sigma}{T}, \quad \lambda = ka, \tag{6}$$

we obtain

$$\eta(r) = \left(\eta_0 e^{-i\varphi_0} - f e^{-i\varphi}\right) \frac{J_0(kr)}{J_0(\lambda)} + f e^{-i\varphi}, \tag{7}$$

where J_n is the n-th order Bessel function.

3. The total sound power

The acoustic power is calculated according to the definition

$$P = 1/2 \int_S p(\mathbf{r}) v^*(\mathbf{r}) dS, \tag{8}$$

where

$$p(\mathbf{r}) = \frac{-ik_0 c\rho_0}{2\pi} \int_{S_0} v(\mathbf{r}_0) \frac{\exp(ik_0 |\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|} dS_0 \tag{9}$$

is the sound pressure , v^* is the conjugate value for the vibration velocity of membrane's points, $|\mathbf{r} - \mathbf{r}_0|$ is the distance from the membrane's point to a point in the soundfield, ρ_0 – rest density of fluid medium, k_0, c correspondingly denote the wave number and propagation velocity, S, S_0 is membrane's area.

The total sound power radiated by some surface sources can be expressed in its Hankel representation

$$P = \pi\rho_0 c k_0^2 \int_0^\infty \frac{W(x)W^*(x) x dx}{\mu}, \tag{10}$$

where

$$W(x) = -i\omega \int_0^a \eta(r) J_0(k_0 x r) r dr \tag{11}$$

is the function characterizing a source, $\mu = \sqrt{1 - x^2}$ for $0 \le x \le 1$, $\mu = i\sqrt{x^2 - 1}$ for $1 \le x \le \infty$ [6, 7].

On the basis of expression (10), the total sound power can be formulated briefly as

$$P = P_{\text{active}} - iP_{\text{reactive}}, \tag{12}$$

where P_{active} and P_{reactive} denote the active and reactive sound power.

Substituting the solution of the equation of motion into Eq. (11) we get as a result

$$W(x) = -i\frac{\omega a^2}{\beta} \left\{ f e^{-i\varphi} \frac{J_1(\beta x)}{x} \right. \left. + (\eta_0 e^{-i\varphi_0} - f e^{-i\varphi}) \frac{\alpha\delta J_0(\beta x) - x J_1(\beta x)}{\delta^2 - x^2} \right\}, \tag{13}$$

where the following notations have been introduced:

$$\beta = k_0 a, \qquad \delta = \frac{\lambda}{\beta}, \qquad \alpha = \frac{J_1(\lambda)}{J_0(\lambda)}. \tag{14}$$

Futher some mathematical calculations lead to

$$W(x)W^*(x) = \frac{\omega^2 a^4 f^2}{\beta^2} \left\{ \frac{J_1^2(\beta x)}{x^2} + \left(1 + q^2 - 2q\cos l\right) \frac{\left[\alpha\delta J_0(\beta x) - xJ_1(\beta x)\right]^2}{\left(\delta^2 - x^2\right)^2} - 2\left(q\cos l - 1\right) \frac{J_1(\beta x)}{x} \, \frac{\alpha\delta J_0(\beta x) - xJ_1(\beta x)}{\delta^2 - x^2} \right\}, \tag{15}$$

where:

$$q = \frac{\eta_0}{f}, \qquad l = \varphi - \varphi_0. \tag{16}$$

Since the above notations being dimensionless, they are very convenient for describing the excitations.

3.1. The active power

The formulation for the active power is derived from formula (10) by performing integration along the real axis within the limits $x \in (0, 1)$ [6]

$$P_{\text{active}} = \pi\rho_0 c k_0^2 \int_0^1 \frac{W(x)W^*(x)x\,dx}{\sqrt{1 - x^2}}. \tag{17}$$

Inserting Eq. (15) into (17) and introducing the following notations:

$$\begin{aligned}
&P_0 = \pi a^2 \rho_0 c \omega^2 f^2, \qquad E_1 = \int_0^1 \frac{J_1^2(\beta x)\,dx}{x\sqrt{1 - x^2}}, \\
&E_2 = \int_0^1 \frac{\left[\delta\alpha J_0(\beta x) - xJ_1(\beta x)\right]^2 x\,dx}{\left(\delta^2 - x^2\right)^2 \sqrt{1 - x^2}}, \\
&E_3 = \int_0^1 \frac{J_1(\beta x)\left(\alpha\delta J_0(\beta x) - xJ_1(\beta x)\right)dx}{\left(x^2 - \delta^2\right)\sqrt{1 - x^2}}, \\
&A = 1 + q^2 - 2q\cos l, \qquad B = 2(q\cos l - 1),
\end{aligned} \tag{18}$$

we transform the formula for active power to the form [4]

$$P_{\text{active}} = P_0 \left(E_1 + AE_2 - BE_3\right), \tag{19}$$

where the magnitude of P_0 is regarded as a unit of the acoustic power.

The integrals given by formulae (18) can be calculated on the basis of the Levine's and Leppington's method and of the Cauchy's theorem of residues. It is necessary for computation to assume that $\delta < 1$.

Computing the integral E_1 we introduce the function of complex variable $z = x+iy$,

$$F_1(z) = \frac{J_1(\beta z)H_1^{(1)}(\beta z)}{z\sqrt{1-z^2}}, \tag{20}$$

which satisfies the following conditions:

$$\text{Re}F_1(x) = \frac{J_1^2(\beta x)}{x\sqrt{1-x^2}}, \qquad \text{Re}F_1(iy) = 0, \tag{21}$$

where by $H_n^{(1)}$ we have denoted the first kind, n-th order Hankel function.

Choosing the suitable, closed path of integration **C** within and along which the considered function is analytical, regular and unique and using the Cauchy's theorem, we obtain [1]

$$\oint\limits_C F_1(z)dz = 0. \tag{22}$$

The obtained equation can be transformed to the symbolic form

$$P\int\limits_0^1 + \int\limits_1^\infty + \int\limits_{R\infty} + \int\limits_{i\infty}^0 = \frac{1}{2}\text{Res}F_1(0),$$

where symbol P denotes that the integral within the limits (0,1) is interpreted as the Cauchy principal value.

The integral computed along the great circle when its radius increases infinitely is equal to zero. The equality (21) results in the fact that there is also no contribution from the integral calculated along the imaginary axes [6]. Finally we get

$$E_1 = \text{Re}\left\{\pi i\frac{1}{2}\text{Res } F_1(0)\right\} + \int\limits_1^\infty \frac{\text{Im}J_1(\beta x)H_1^{(1)}(\beta x)}{x\sqrt{x^2-1}}dx. \tag{23}$$

The integral within the limits $(1,\infty)$ can be evaluated using the stationary phase method and the asymptotes for Bessel functions.

Taking into consideration that

$$\text{Res}F_1(0) = -\frac{i}{\pi}, \tag{24}$$

we obtain

$$E_1 = \frac{1}{2}\left(1 + \frac{\cos\gamma}{\sqrt{\beta\pi}\beta}\right), \tag{25}$$

where the following notation has been introduced:

$$\gamma = 2\beta + \pi/4. \tag{26}$$

The integrals E_2 and E_3 can be calculated in an analogous way by choosing the suitable functions and path of integration [1]. For integral E_2 the following function is introduced:

$$\widetilde{F}_2 = \frac{zF_2(z)}{(\delta^2 - z^2)^2\sqrt{1-z^2}}, \tag{27}$$

where

$$F_2(z) = \alpha^2\delta^2 J_0(\beta z)H_0^{(1)}(\beta z) + z^2 J_1(\beta z)H_1^{(1)}(\beta z) - \alpha\delta z\left[J_0(\beta z)H_1^{(1)}(\beta z) + J_1(\beta z)H_0^{(1)}(\beta z)\right]. \tag{28}$$

Basing on the Cauchy's theorem concerning the residues and on the following relations:

$$\frac{\mathrm{Im}F_2(\delta)}{\delta^2} = 0, \qquad \mathrm{Im}\left(\frac{dF_2(\delta)}{dz}\right) = -\frac{2\delta^2\beta}{\pi\lambda}\left(1+\alpha^2\right), \tag{29}$$

we obtain the final result

$$E_2 = \frac{1}{2}\left\{\frac{1+\alpha^2}{\sqrt{1-\delta^2}} + \frac{(1-\alpha^2\delta^2)\cos\gamma + 2\delta\alpha\sin\gamma}{\sqrt{\beta\pi}\beta(1-\delta^2)}\right\}. \tag{30}$$

To evaluate the integral E_3, the function is introduced in the form

$$\widetilde{F}_3(z) = \frac{F_3(z)}{(z^2-\delta^2)\sqrt{1-z^2}}, \tag{31}$$

where

$$F_3(z) = \frac{1}{2}\alpha\delta\left[J_1(\beta z)H_0^{(1)}(\beta z) + J_0(\beta z)H_1^{(1)}(\beta z)\right] - zJ_1(\beta z)H_1^{(1)}(\beta z). \tag{32}$$

In this case the path of integration is the same as that used in integral E_2.

Some mathematical calculations lead to

$$E_3 = \frac{\alpha}{\lambda\sqrt{1-\delta^2}} + \frac{1}{\sqrt{\beta\pi}\beta\,(1-\delta^2)}\left[\frac{\lambda\alpha}{\beta}\sin\gamma + \cos\gamma\right]. \tag{33}$$

The formula (19) together with the formulae (25), (30) and (33) represent the active sound power radiated by the investigated source.

3.2. The reactive power

The reactive power can be derived from the integral formulation [8]

$$P_{\text{reactive}} = \pi\rho_0 c k_0^2 \int_1^\infty \frac{W(x)W^*(x)x\,dx}{\sqrt{x^2-1}}. \tag{34}$$

Computing the above integral we use the stationary phase method and the following asymptotic formulae for Bessel functions:

$$\begin{aligned} &J_0^2(\beta x) \approx \frac{1}{\pi\beta x}\left\{1+\sin 2\beta x\right\}, \qquad J_1^2(\beta x) \approx \frac{1}{\pi\beta x}\left\{1-\sin 2\beta x\right\}, \\ &J_0(\beta x)J_1(\beta x) \approx -\frac{1}{\pi\beta x}\cos 2\beta x. \end{aligned} \tag{35}$$

After some mathematical transformations, the reactive power according to the Eqs. (18), (26) can be expressed in the final form

$$\begin{aligned} P_{\text{reactive}} = \frac{P_0}{\pi\beta}\Bigg\{ f^2\left(1-\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\sin\gamma\right) + \left[A\frac{1-\alpha^2(1-2\delta^2)}{2(1-\delta^2)}+B\right]\Omega \\ + \frac{1}{2(1-\delta^2)}\sqrt{\frac{\pi}{\beta}}\Bigg[\left(\frac{4A(\delta^2\alpha^2-1)}{1-\delta^2}-B\right)\sin\gamma \\ + \alpha\delta\left(\frac{8A}{1-\delta^2}+B\right)\cos\gamma\Bigg]\Bigg\}, \end{aligned} \tag{36}$$

where

$$\Omega = \frac{\arcsin\delta}{\delta\sqrt{1-\delta^2}}. \tag{37}$$

The active and reactive sound power is expressed by the elementary formulae which are convenient for further numerical calculations.

4. Conclusions

On the basis of the asymptotic formulae for the acoustic power and of the more general integral formulation (17), (34) we come to a conclusion that both the active and reactive sound power of the considered source for arbitrary values of parameters: q, β, λ reach maximum value when the difference of phases between excitation equals π (Fig. 3, 4). That is when a transverse deflection of points on the surface of the membrane and the displacement at the edge have opposite directions.

For some nearby resonance frequencies we observe a distinct increase of the acoustic power (Fig. 5). Neglecting of damping causes that the sound power increases infinitely when frequency of the excitations tends to the resonance value.

The acoustic sound power radiated by a source can be controlled by means of selection of appropriate amplitudes, phases and frequency characterizing both kinds of excitations. The derived formulae, having elementary form, are very useful for some numerical calculations.

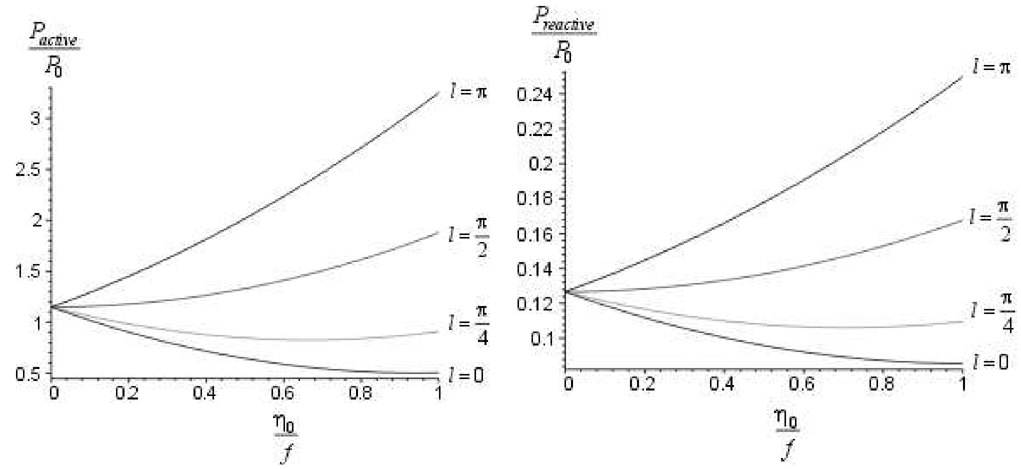

Fig. 2. The active and reactive sound power plotted in terms of parameter η_0/f for $\beta = 15$, $\delta = 0.5$ and for some values of the differences between the phases associated with the excitations.

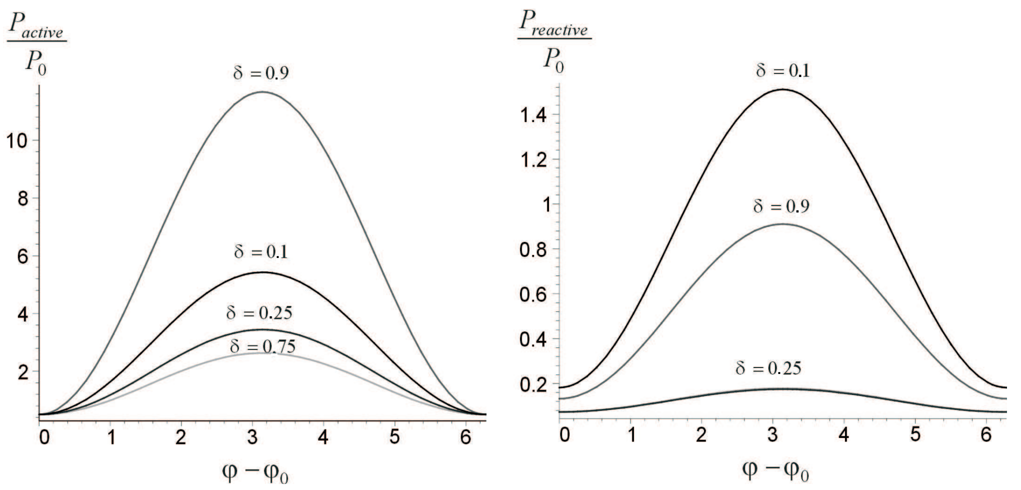

Fig. 3. The active and reactive sound power in terms of parameter $\varphi - \varphi_0$ in the case of $\eta_0/f = 1$, $\beta = 15$.

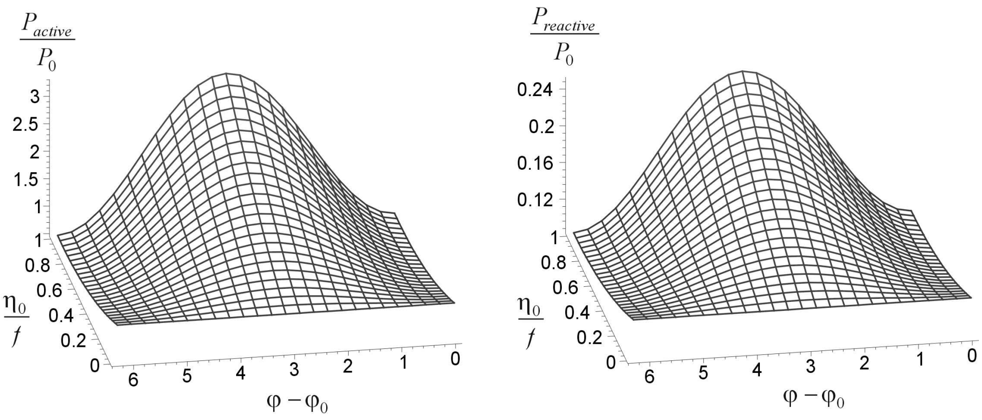

Fig. 4. The active and reactive power plotted as a function of parameters η_0/f and $\varphi - \varphi_0$ for $\beta = 15$ and $\delta = 0.5$.

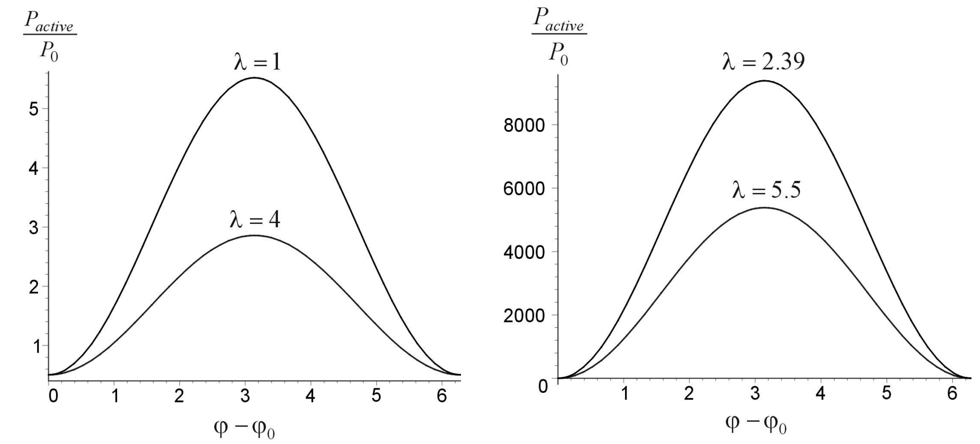

Fig. 5. The active acoustic power in terms of $\varphi - \varphi_0$ for frequencies correspondingly near and far from the resonance when $\beta = 15$, $\eta_0/f = 1$.

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