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NONLINEAR REFLECTION AND TRANSMISSION OF PLANE ACOUSTIC WAVES

J. WÓJCIK

Institute of Fundamental Technological Research, Polish Academy of Sciences
Department of Ultrasound
Świętokrzyska 21, 00-049 Warszawa, Poland
e-mail: jwojcik@ippt.gov.pl

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In the present, paper the classical problem of reflection and transmission of a plane acoustic wave is analyzed and solved for nonlinear propagation. Two adjacent media with a plane boundary between them are assumed. The parameters characterizing the properties of the media can be changed stepwise on the boundary. The wave incident on the boundary surface is plane. It was assumed that the disturbance in the first medium is a superposition of the incident and reflected waves, and in the second medium there is only the transmitted wave. On the base of nonlinear acoustic equations, assuming continuity of the velocity and pressure fields, the reflection and transmission operators of velocities and pressures were determined. The operators are nonlinear in relation to the incident wave field. It was found that near the boundary there occurs "a reflecting-transmitting" layer which is decisive for the description of the nonlinear phenomenon of the reflection and transmission. There arises a nonlinear feedback between the reflecting and incident waves. This is the fundamental difference between the nonlinear and the linear reflection. Equations of the incident reflected and transmitted waves are given. In the case of classical viscous media, they are the Burger's equations in asymptotic areas. The operators and the experimental significance of the results obtained were additionally discussed. An example of the effective application of the analysis performed is given in Sec. 6.

Key words: reflection, transmission, nonlinearity.

1. Introduction

The reflection and transmission of acoustic disturbances is a fundamental problem of the propagation theory in inhomogeneous media. Even in the linear propagation case beyond the known classical solutions, the problems which we meet here are difficult and the corresponding literature is extensive [1]. On the contrary, in the case of a nonlinear description of this problem, the acoustic literature is very scant. A few theoretical and experimental papers (Refs. [2–5]) are worthy of notice. They encouraged the author to perform this study. A solution of the fundamental nonlinear problem, i.e. how to determine the reflected and transmitted disturbances as a function of the incident disturbance in the shape of the plane wave incident normally on the plane boundary between two media has not found in the literature. In other words, to determine nonlinear quantities corresponding to the reflection and transmission coefficients known from the linear propagation theory. Their shape results from the propagation equation and from the general continuity conditions of the fields on the boundary. The proper number of equations and unknown variables cause that they can not be a subject of definition or can not be constructed in the way of generalization of ideas (for instance impedances) known from linear theory. The phenomenon, which will be considered in this paper, is a part – often a fundamental one – of many technologies. For example, in medical ultrasonic diagnostic methods important information is obtained due to the detection of reflections from boundaries between different tissues. Often equilibrium parameters (impedances) of tissues differ from each other by a very small amount. It arises a question – which may be important not only for medical diagnostics [4, 5] – how the nonlinear effects influence the general picture of the reflection and transmission phenomenon? The nonlinearity parameter B/A for different soft tissues can vary from 5.8 (cardiac muscle) to 11 (fatty tissue) [6]. Can the media, which differ only by this parameter, be differentiated from each other due to different reflections? What is the qualitative and quantitative effect of this phenomenon? The aim of this paper is to find a response to the aforementioned question in its simplest dimensional geometrical configuration without taking into consideration any transverse disturbances, which may occur in relation to the beam axis.

2. Formulation of the problem

Two adjacent media with a plane interface between them are considered. Parameters and material function characteristics of the first (left) medium and the second one will be denoted by either top or bottom indices $m = 1,\ 2$, respectively (Fig. 1). In our analysis

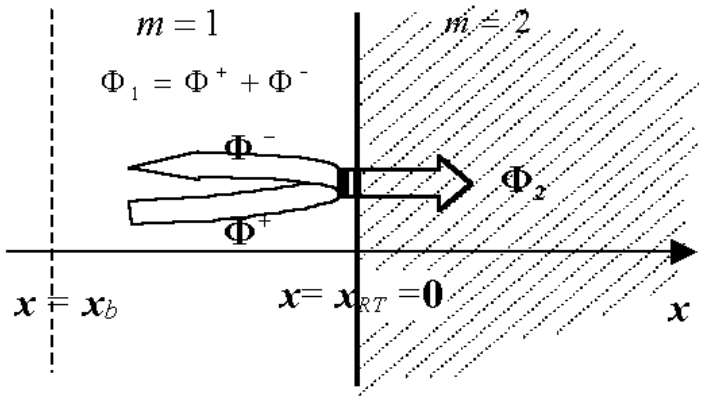

Fig. 1. The incident Φ^+, reflected Φ^-, and transmitted Φ_2 waves near the interface situated at $x = x_{RT} = 0$ between the media $m = 1$ and $m = 2$.

a dimensionless system of variables and parameters will be used. The dimensionless equilibrium density, g_{0m}, and sound speed, c_{m0}, are given by the relations

$$g_{0m} = \rho_{0m}/\rho_0, \qquad c_{m0} = c_{0m}/c_0, \tag{1}$$

where ρ_{0m}, c_{0m} are the equilibrium density and velocity of sound for the m-th medium; ρ_0, c_0 are arbitrary quantities corresponding to the density and velocity of sound (of course, it can be $\rho_0 = \rho_{01}$, $c_0 = c_{01}$).

It is supposed that the media are lossy and nonlinear with respect to disturbance. The disturbance in the first medium will be described by the acoustical potential Φ_1, and in the second one by Φ_2. The following representations of the fields in media will be used:

$$\Phi_1(x,t) = \Phi^+\left(t - \frac{x}{c_{10}}, x\right) + \Phi^-\left(t + \frac{x}{c_{10}}, -x\right), \qquad x \le x_{RT}, \quad m = 1, \tag{2}$$

$$\Phi_2(x,t) = \Phi_2\left(t - \frac{x}{c_{20}}, x\right), \qquad x \ge x_{RT}, \quad m = 2, \tag{3}$$

where $\Phi^+ + \Phi^-$ denotes the composition of the incident and reflected waves; (x,t) are the space and time coordinates; x_{RT} denotes the space coordinate of the boundary plane (Fig. 1). From the definition $\mathbf{v} \equiv \nabla\Phi$ (∇ – denotes here the gradient), this means that the velocity field is linear with respect to the potential. Therefore, the following decompositions of the velocity, corresponding to (2), (3), are valid

$$\begin{aligned} v_1(x,t) &= v^+\left(t - \frac{x}{c_{10}}, x\right) + v^-\left(t + \frac{x}{c_{10}}, -x\right), \\ v_1 &= \mathbf{e}\cdot\mathbf{v}_1 = \mathbf{e}\cdot\mathbf{v}^+ + \mathbf{e}\cdot\mathbf{v}^-, \qquad \mathbf{v}_1 = \mathbf{e}\,v_1\,, \end{aligned} \tag{4}$$

$$\begin{aligned} v_2(x,t) &= v_2\left(t - \frac{x}{c_{20}}, x\right), \\ v_2 &= \mathbf{e}\cdot\mathbf{v}_2\,, \qquad \mathbf{v}_2 = \mathbf{e}\,v_2\,, \end{aligned} \tag{5}$$

where $\mathbf{e}$ is the unit vector in the positive direction of the x-axis.

Similarly, the following decompositions of the acoustical pressure are applied

$$P_1(x,t) = P^+\left(t - \frac{x}{c_{10}}, x\right) + P^-\left(t + \frac{x}{c_{10}}, -x\right), \tag{6}$$

$$P_2(x,t) = P_2\left(t - \frac{x}{c_{20}}, x\right). \tag{7}$$

Precisely speaking, the relation between the acoustical pressure and the potential follows from the definition $\mathbf{v} \equiv \nabla\Phi$ and from the general equations of motion of a continuous medium, and is not linear:

$$P_1 = \mathsf{P}_1\left[\Phi_1\right] = \mathsf{P}_1\left[\Phi^+ + \Phi^-\right] \neq \mathsf{P}_1\left[\Phi^+\right] + \mathsf{P}_1\left[\Phi^-\right] = P^+ + P^-. \tag{8}$$

In Appendix A1, the definition of the nonlinear operator $\mathsf{P}_m\left[\,\cdot\,\right]$, and the justification of the very good approximation of $\mathsf{P}_m \cong -g_{0m}\partial_t$ (Eqs. (A10), (A12)) are given. Nevertheless, we would like to stress that the form of $\mathsf{P}_m\left[\,\cdot\,\right]$ has no significant influence on our considerations.

Firstly, we would like to determine the relations between v^+, v^-, v_2 and P^+, P^-, P_2 on the interface at $x = x_{RT} = 0$. Then we will research the boundary conditions for v^-, v_2 and P^-, P_2 on the plane between the two different nonlinear and lossy media. This problem can be solved if we find the functions R, T (operators) such that,

$$v^- = \mathsf{R}'_v\left[\{m\}\,;v^+\right] = \mathsf{R}_v\left[\{m\}\,;v^+\right]\circ v^+, \tag{9}$$

$$v_2 = \mathsf{T}'_v\left[\{m\}\,;v^+\right] = \mathsf{T}_v\left[\{m\}\,;v^+\right]\circ v^+, \qquad x = x_{RT} = 0, \tag{10}$$

where $\mathsf{R}_v\left[\,\cdot\,;\,\cdot\,\right]$, $\mathsf{T}_v\left[\,\cdot\,;\,\cdot\,\right]$ are the reflection and transmission operators; in the general case $\circ$ – denote operation characteristics for the operator. Generally, in almost all the considered cases $\circ \equiv \otimes$ is a convolution (in the time or Fourier frequency domain), but sometimes, in special cases, $\circ \equiv \cdot$ is an ordinary multiplication; $\{m\}$ denotes a set of material parameters which characterize the media.

Secondly, we would like to find equations which describe the evolutions of the disturbances Φ^+, Φ^- and Φ_2.

In the linear case, we have the "classical" problem of reflection and transmission of a plane wave. In the Fourier frequency domain

$$\hat{v}^- = \widehat{\mathsf{R}}_v\left[\{m\}\right]\circ \hat{v}^+ = \widehat{R}_v\left[\{m\}\right]\cdot \hat{v}^+, \tag{11}$$

$$\hat{v}_2 = \widehat{\mathsf{T}}_v\left[\{m\}\right]\circ v^+ = \widehat{T}_v\left[\{m\}\right]\cdot \hat{v}^+, \tag{12}$$

$\widehat{R}\left[\{m\}\right]$, $\widehat{T}\left[\{m\}\right]$ are the reflection and transmission coefficients.

3. Basic equations

As the base of our description we assume Eq. (22) referred in [7]. It describes finite amplitude potential disturbances in lossy media. In the Cartesian coordinate system and for one-dimensional disturbances in the m-th medium, it takes the form

$$c_{m0}^2\partial_{xx}\Phi_m - \partial_{tt}\Phi_m - 2\mathcal{A}_m\partial_t\Phi_m - q_m\partial_t\left(\partial_t\Phi_m\right)^2 = 0 + O\big((q+\alpha)^2\big), \tag{13}$$

$$q_m \equiv q(\gamma_m+1)/2c_{m0}^2 = q\beta_m/c_{m0}^2, \qquad m = 1,2, \tag{14}$$

where γ_m is either the exponent of the adiabate or $\gamma_m = (B/A)_m + 1$, $(B/A)_m$ is the nonlinearity parameter; $q \equiv P_0/\rho_0 c_0^2$; P_0 is the characteristic pressure (i.e. the pressure amplitude of the disturbance); $O\big((\cdot)^l\big)$ is a small quantity of the order of l. $\mathcal{A}_m$ is the convolution type operator of absorption. In this paper its representation in the time domain is accepted in the following form:

$$\mathcal{A}\Phi \equiv A(t)\otimes\Phi(x,t), \tag{15}$$

$$A(t) = F^{-1}\left[a(\omega)\right], \tag{16}$$

where $a(\omega)$ is the small signal coefficient of absorption (eigenvalue of $\mathcal{A}$, corresponding to a disturbance in the form of eigenfunction, a Fourier function with the frequency ω); $F[\,\cdot\,]$ is the Fourier transformation. For the classical absorption (for the m-th medium) $a^m(\omega) = \alpha_2^m \omega^2$, where α_2^m is the dimensionless hybrid viscosity, $\mathcal{A}_m = -\alpha_2^m \partial_{tt} + O(\alpha(q+\alpha))$. For the details of normalization and the descriptions of Eq. (13) and the absorption operator $\mathcal{A}$ see Ref. [7].

We introduce

$$\tau^+(t,x) = t - \frac{x}{c_{10}}, \qquad \tau^-(t,x) = t + \frac{x}{c_{10}}, \qquad x \le 0, \tag{17}$$

$$\tau_2(t,x) = t - \frac{x}{c_{20}}, \qquad x \ge 0. \tag{18}$$

It should be stressed that we may also apply the following arguments for Φ^+, Φ^- and Φ_2:

$$\tau^+(t,x) = t - \frac{|x - x_b|}{c_{10}} = t - \frac{x - x_b}{c_{10}}, \qquad x_b \le x \le 0, \tag{19}$$

$$\tau^-(t,x) = t - \frac{|x|}{c_{10}} = t + \frac{x}{c_{10}}, \qquad x \le 0, \tag{20}$$

$$\tau_2(t,x) = t - \frac{x}{c_{20}}, \qquad x \ge 0, \tag{21}$$

where x_b is an arbitrary value or coordinate of the plane in which time profile of Φ^+ is given: $\Phi^+(t,x)\Big|_{x=x_b} = \Phi_b^+(t)$. The arguments in the form of (19), (20), (21) (after the first sign of equality) are evidently retarded in time and more adequate for the more complex problem, i.e. for the boundary value problem at the plane $x = x_b$ for propagation to the reflected and transmitted plane at $x = x_{RT} = 0$. Nevertheless, both forms give the same results in our problem. We search a solutions of Eq. (13) in the form (2) in the first medium $m = 1$, and in the form (3) in the second one $m = 2$.

After substitution $\Phi_2 = \Phi_2(t - x/c_{20}, x) = \Phi_2(\tau_2, x)$ in Eq. (13), we have

$$c_{20}^2 \left[\partial_{xx}\Phi_2(\tau_2,x) - \frac{2}{c_{20}} \partial_x \partial_{\tau_2} \Phi_2(\tau_2,x) \right] = 2\partial_{\tau_2} \mathcal{A}_2 \Phi_2(\tau_2,x) + q_2 \partial_{\tau_2} \left(\partial_{\tau_2}\Phi_2(\tau_2,x)\right)^2, \tag{22}$$

$$(\partial_t - \partial_{\tau_2})\Phi_2 = 0, \tag{23}$$

where the differentiation with respect to x concerns now only the second argument in Φ_2. Formally, the fully variable transformation includes the transformation $x \to \xi = x$. However, in order to limit the number of symbols, we will use the old ones. The functions $\tau^{\pm,2}(t,x) = (t \mp x/c_{1,20})$ are a pair of characteristics of the d'Alambertian operator $\Box \equiv c_{m0}^2 \partial_{xx} - \partial_{tt}$. On the characteristics, the solutions $\Phi(t \mp x/c_{m0})$ of the equation $\Box\,\Phi = 0$ are constants. This means physically a mutual compensation of the fast space changes (in the $\lambda_0 \equiv 2\pi/k_0$ scale) with changes in time of the disturbances (in the scale $T_0 \equiv 2\pi/\omega_0$, $\omega_0 = c_{m0}k_0$). In characteristic coordinates $\partial_x(\Phi(\tau^{\pm,2})) = 0$.

In the solutions of the equations disturbed by absorption or a nonlinear term of the $O(q+\alpha)$, an additional dependence on the coordinate (in our case x) occurs in the characteristic space scale $\lambda_{\alpha,q} \equiv \min(1/\alpha, 1/q)$. The relation $\partial_x(\partial_x \Phi_2, \partial_{\tau_2}\Phi_2) = O(q+\alpha)$ follows from (22). However, $\partial_{xx}\Phi_2 = O((q+\alpha)^2)$. The proof of this relation is given in the Appendix A2. The terms of $O\big((q+\alpha)^2\big)$ were already neglected in (13). We neglect them also in (22). We have

$$\partial_x\partial_t\Phi_2(\tau_2,x) = -\frac{1}{c_{20}}\mathcal{A}_2\partial_t\Phi_2(\tau_2,x) - \frac{q_2}{2c_{20}}\partial_t\left(\partial_t\Phi_2(\tau_2,x)\right)^2, \qquad \partial_{\tau_2} = \partial_t\,. \tag{24}$$

Supposing zero initial conditions in the neighborhood of $x = x_{RT} = 0$ and integrating (24) over the time, we have

$$\partial_x\Phi_2(\tau_2,x) = -\frac{1}{c_{20}}\overline{\mathcal{A}}_2\partial_{\tau_2}\Phi_2(\tau_2,x) - \frac{q_2}{2c_{20}}\left(\partial_{\tau_2}\Phi_2(\tau_2,x)\right)^2, \tag{25}$$

where $\overline{\mathcal{A}}_m \equiv \int \mathcal{A}_m$, $\int\limits_0^t \mathcal{A}_m\partial_{t'}\Phi\, dt' = \mathcal{A}_m\Phi = \overline{\mathcal{A}}_m\partial_t\Phi$. On the basis of (15) and (16), we have $\overline{\mathcal{A}}_m\partial_t\Phi = \overline{A}_m(t)\otimes(\partial_t\Phi)$, $\overline{A}_m(t) = \int\limits_0^t A_m(t')dt'$, $F\left[\overline{A}_m\right] = \overline{a}^m(\omega) = a^m(\omega)/(-i\omega)$. For a classically absorbing medium $\overline{\mathcal{A}}_m = -\alpha_2^m\partial_t$.

Applying the aforementioned operations to the $\Phi_1 = \Phi^+ + \Phi^-$, we obtain

$$\begin{aligned}\partial_t\partial_x\big(\Phi^+(\tau^+,x) - \Phi^-(\tau^-,-x)\big) \;&=\; -\frac{1}{c_{10}}\mathcal{A}_1\partial_t\left(\Phi^+(\tau^+,x) + \Phi^-(\tau^-,-x)\right)\\ &\quad - \frac{q_1}{2c_{10}}\partial_t\left(\partial_t\Phi^+(\tau^+,x) + \partial_t\Phi^-(\tau^-,-x)\right)^2,\end{aligned} \tag{26}$$

and

$$\begin{aligned}\partial_x\big(\Phi^+(\tau^+,x) - \Phi^-(\tau^-,-x)\big) \;&=\; -\frac{1}{c_{10}}\overline{\mathcal{A}}_1\big(\partial_{\tau^+}\Phi^+(\tau^+,x) + \partial_{\tau^-}\Phi^-(\tau^-,-x)\big)\\ &\quad - \frac{q_1}{2c_{10}}\left(\partial_{\tau^+}\Phi^+(\tau^+,x) + \partial_{\tau^-}\Phi^-(\tau^-,-x)\right)^2,\end{aligned} \tag{27}$$

$$\left(\partial_{\tau^+} - \partial_t\right)\Phi^+ = 0, \qquad \left(\partial_{\tau^-} - \partial_t\right)\Phi^- = 0. \tag{28}$$

Equations (24), (25) and (26), (27) are equivalent to (13) in the medium $m = 2$ and $m = 1$, respectively, with accuracy to the terms of $O(\alpha(q+\alpha))$. This means that if Φ_2 satisfies (24), and Φ^+ and Φ^- satisfy (26), then $\Phi_1 = \Phi^+ + \Phi^-$ and Φ_2 satisfy (13). With zero initial conditions for the potentials and their first derivatives, Eqs. (24), (26) with respect to time are equivalent to (25), (27). However, if the functions Φ_2, Φ^+, Φ^- satisfy (25), (27) then they also satisfy (24) and (26).

For assumed shapes of the solutions Φ^+, Φ^-, Φ_2, the adequate acoustical pressures and velocities take the form,

$$P^+ = -g_{01}\partial_t\Phi^+ = -g_{01}\partial_{\tau^+}\Phi^+, \tag{29}$$

$$P^- = -g_{01}\partial_t\Phi^- = -g_{01}\partial_{\tau^-}\Phi^-, \tag{30}$$

$$P_2 = -g_{02}\partial_t\Phi_2 = -g_{02}\partial_{\tau_2}\Phi_2, \tag{31}$$

$$v^+ = -\frac{1}{c_{10}}\partial_t\Phi^+ + \partial_x\Phi^+ = -\frac{1}{c_{10}}\partial_{\tau^+}\Phi^+ + \partial_x\Phi^+ = \frac{P^+}{z_{01}} + \partial_x\Phi^+, \tag{32}$$

$$v^- = \frac{1}{c_{10}}\partial_t\Phi^- + \partial_x\Phi^- = \frac{1}{c_{10}}\partial_{\tau^-}\Phi^- + \partial_x\Phi^- = -\frac{P^-}{z_{01}} + \partial_x\Phi^-, \tag{33}$$

$$v_2 = -\frac{1}{c_{20}}\partial_t\Phi_2 + \partial_x\Phi_2 = -\frac{1}{c_{20}}\partial_{\tau_2}\Phi_2 + \partial_x\Phi_2 = \frac{P_2}{z_{02}} + \partial_x\Phi_2\,, \tag{34}$$

where $z_{0m} \equiv g_{0m}c_{m0}$, are the equilibrium impedances. From Eqs. (32)–(34) we obtain

$$P^+ = z_{01}(v^+ - \partial_x\Phi^+), \tag{35}$$

$$P^- = -z_{01}(v^- - \partial_x\Phi^-), \tag{36}$$

$$P_2 = z_{02}(v_2 - \partial_x\Phi_2). \tag{37}$$

The functions $\partial_x\Phi^+$, $\partial_x\Phi^-$, $\partial_x\Phi_2$ show the substantial differences between the impedance relations for an ideal linear medium and for a lossy or nonlinear one.

Taking into account the above formulas, we see that Eq. (24) is the Burger's equation for P_2 (which shall be called "generalized Burger's" equations due to the generalization of the description of absorption [7, 11, 12]). Equation (26) may by interpreted as the nonlinear coupled by means of the term $(q_1/z_{01})\,\partial_t\,(P^+P^-)$ of two Burger's equations for P^+ and P^-.

4. Continuity conditions and equations

In this and the next section, all the functions and relations on the *interface* are considered and analyzed. On this surface at $x = x_{RT} = 0$ all the functions and relations depend only on time t, and $\partial_t = \partial_{\tau^+} = \partial_{\tau^-} = \partial_{\tau_2}$. We also preserve the prevailing arrangement of signs.

We assume continuity conditions in the conventional form

$$P_1 = P_2\,, \tag{38}$$

$$v_1 = v_2\,, \qquad \text{for } x = x_{RT} = 0. \tag{39}$$

On the basis of Eq. (A12), we can use in (38) the decomposition shown in (6) and (7). Then we apply (35)–(37) to reduce P^+, P^-, P_2 from (38). The decompositions (4), (5) are inserted to (39). After the aforementioned operations, Eqs. (38) and (39) take the form

$$z_{01}\left((v^+ - v^-) - \partial_x(\Phi^+ - \Phi^-)\right) = z_{02}\left(v_2 - \partial_x\Phi_2\right), \tag{40}$$

$$v^+ + v^- = v_2\,. \tag{41}$$

The needed terms $\partial_x(\Phi^+ - \Phi^-)$, $\partial_x\Phi_2$ were obtained from (25), (27) in the limiting transition process $x \to \left(x^{\leq}_{RT},\, x^{\geq}_{RT}\right)$, $x = x_{RT}$,

$$\left(\partial_x\Phi^+ - \partial_x\Phi^-\right) = \frac{1}{z_{01}}\overline{\mathcal{A}}_1\left(P^+ + P^-\right) - \frac{q_1c_{10}}{2z_{01}^2}\left(P^+ + P^-\right)^2 + O((q+\alpha)^2), \tag{42}$$

$$\partial_x\Phi_2 = \frac{1}{z_{02}}\overline{\mathcal{A}}_2P_2 - \frac{q_2c_{20}}{2z_{02}^2}P_2^2 + O((q+\alpha)^2). \tag{43}$$

On the basis of Eqs. (35)–(37), we obtain with the same accuracy

$$\left(\partial_x\Phi^+ - \partial_x\Phi^-\right) = \overline{\mathcal{A}}_1\left(v^+ - v^-\right) - \frac{q_1c_{10}}{2}\left(v^+ - v^-\right)^2 + O((q+\alpha)^2), \tag{44}$$

$$\partial_x\Phi_2 = \overline{\mathcal{A}}_2v_2 - \frac{q_2c_{20}}{2}v_2^2 + O((q+\alpha)^2). \tag{45}$$

Each of the Eqs. (26), (27), (42), (44) will be called "couple equation".

Substituting (44), (45) into (40) and using (41) we obtain

$$\left[\mathsf{z}_1 + \mathsf{z}_2 + w\cdot v^+\right]v^- = \left[\mathsf{z}_1 - \mathsf{z}_2 + \frac{1}{2}u\cdot v^+\right]v^+ + \frac{1}{2}u\cdot(v^-)^2, \tag{46}$$

where z_1, z_2 are operators of the linear impedance

$$\mathsf{z}_m \equiv z_{0m}\left(1 - \overline{\mathcal{A}}_m\right), \qquad m = 1, 2, \tag{47}$$

$$w \equiv z_{01}c_{10}q_1 + z_{02}c_{20}q_2\,, \tag{48}$$

$$u \equiv z_{01}c_{10}q_1 - z_{02}c_{20}q_2\,. \tag{49}$$

Generally, Eq. (46) (due to absorption) may be an integral-differential nonlinear equation in the time domain, or a nonlinear convolution equation in the Fourier frequency domain). The simplest case with regard to the absorption is obtained when the classical absorption $\overline{\mathcal{A}}_m = -\alpha_2^m\partial_t$ is assumed for both the media. In this case (46) is the Riccati equation.

5. Reflection and transmissions operators

The abstract (symbolical) solution of Eq. (46) can be presented as follows

$$v^-(t) = \mathsf{R}_v\left[\{m\}; v^+\right]v^+(t), \tag{50}$$

$$\mathsf{R}_v\left[\{m\}; v^+\right] \equiv \frac{1}{2\mathsf{w}\circ\mathsf{R}_0v^+}\left[1 - \sqrt{1 - 4\mathsf{w}\circ\mathsf{R}_0\,v^+}\right]\mathsf{R}_0\,, \tag{51}$$

where $\mathsf{w} = \mathsf{w}\left[\{m\}; v^+\right]$, $\mathsf{R}_0 = \mathsf{R}_0\left[\{m\}; v^+\right]$ are nonlinear operators with respect to v^+,

$$\mathsf{w}\left[\{m\}; v^+\right] \equiv \frac{\frac{1}{2}u\cdot}{\mathsf{z}_1 + \mathsf{z}_2 + w \cdot v^+}, \tag{52}$$

$$\mathsf{R}_0\left[\{m\}; v^+\right] \equiv \frac{\mathsf{z}_1 - \mathsf{z}_2 + \frac{1}{2}uv^+}{\mathsf{z}_1 + \mathsf{z}_2 + wv^+}. \tag{53}$$

The sign + before the square root in definition (51) was neglected since it gives an exotic solution, which does not have any known linear asymptotic form (when $q_m \to 0$). Because $w \leq O(q)$, then the square root in (51) may be written in the form of power series from which we obtain

$$\mathsf{R}_v\left[\{m\}; v^+\right] = \left[1 + \mathsf{w} \circ (\mathsf{R}_0 v^+) \cdot + ...\right] \mathsf{R}_0, \tag{54}$$

$$\mathsf{T}_v\left[\{m\}; v^+\right] = 1 + \mathsf{R}_v\left[\{m\}; v^+\right]. \tag{55}$$

The expansion (54) can be also obtained by applying the successive approximation method to (46). The example of factorization of the operators $\mathsf{w}[\,\cdot\,]$ and $\mathsf{R}_0[\,\cdot\,]$ is presented in the Appendix A3. Assuming the absence of absorption $\mathsf{z}_m = \mathsf{z}_{0m}$ (or the negligibility of it), the simplest case of the factorization is obtained. In this case $\mathsf{w}[\,\cdot\,]$ and $\mathsf{R}_0[\,\cdot\,]$ factorize themselves in ordinary functions, and the formulas of (50), (51), (54) give the factorized solution, $\circ = \cdot$ in the time domain

$$v^-(t) = \left[\left(1 + \frac{\frac{1}{2}u\left(z_{01} - z_{02} + \frac{1}{2}uv^+(t)\right)}{\left(z_{01} + z_{02} + wv^+(t)\right)^2} v^+(t) + ...\right) \cdot \frac{z_{01} - z_{02} + \frac{1}{2}uv^+(t)}{z_{01} + z_{02} + wv^+(t)}\right] v^+(t). \tag{56}$$

Expanding the above formula or the formulas $\mathsf{w}[\cdot]$ and $\mathsf{R}_0[\cdot]$ under conditions $\mathsf{z}_m = z_{0m}$ with respect to q and remaining only the terms of $O(1)$ and $O(q)$, we obtain

$$v^-(t) = \left(\frac{z_{01} - z_{02}}{z_{01} + z_{02}} + \frac{2z_{01}z_{02}q}{\left(z_{01} + z_{02}\right)^3}\left(\frac{z_{02}}{c_{10}}\beta_1 - \frac{z_{01}}{c_{20}}\beta_2\right) v^+(t)\right) v^+(t) + O(q^2), \tag{57}$$

$$v_2(t) = \left(\frac{2z_{01}}{z_{01} + z_{02}} + \frac{2z_{01}z_{02}q}{\left(z_{01} + z_{02}\right)^3}\left(\frac{z_{02}}{c_{10}}\beta_1 - \frac{z_{01}}{c_{20}}\beta_2\right) v^+(t)\right) v^+(t) + O(q^2). \tag{58}$$

We rewrite the above formulas in the form

$$v^- = \mathsf{R}_v v^+ = \left(R_v + r_v v^+\right) v^+, \tag{59}$$

$$R_v \equiv \frac{z_{01}-z_{02}}{z_{01}+z_{02}}\,, \qquad r_v \equiv \frac{q2z_{01}z_{02}}{\left(z_{01}+z_{02}\right)^3}\left(\frac{z_{02}}{c_{10}}\beta_1-\frac{z_{01}}{c_{20}}\beta_2\right),$$
$$v_2=\mathsf{T}_v v^+=\left(T_v+r_v v^+\right)v^+, \tag{60}$$
$$\mathsf{T}_v=1+\mathsf{R}_v\,, \qquad T_v=1+R_v=\frac{2z_{01}}{z_{01}+z_{02}}\,.$$

The knowledge of $\partial_x\left(\Phi^+ + \Phi^-\right)$ as a function of P^+, P^- is required to determine the reflection and transmission operators for pressure on the basis of the continuity conditions (38), (39). Below we present a particular method, which allows the determination of the quantities mentioned. The general method will be presented in the next section.

To determine the reflection – R_p, and transmission – T_p operators for pressure we suppose that:

a)

$$P^-=\mathsf{R}_p P^+, \qquad P_2=\mathsf{T}_p P^+, \tag{61}$$

where

$$\mathsf{R}_p \;=\; R_p-r_pP^+=-\left(R_v+r_pP^+\right), \tag{62}$$
$$\mathsf{T}_p \;=\; 1+\mathsf{R}_p=T_p-r_pP^+=T_v\frac{z_{02}}{z_{01}}-r_pP^+; \tag{63}$$

b) the principle of conservation of energy on the boundary surface $x=x_{TR}=0$ is intact in the form

$$\widetilde{\mathbf{I}}^+ + \widetilde{\mathbf{I}}^- = \widetilde{\mathbf{I}}_2\,, \tag{64}$$

where $\widetilde{\mathbf{I}}^+$, $\widetilde{\mathbf{I}}^-$, $\widetilde{\mathbf{I}}_2$ are the energy current density vectors [7] (the instantaneous value of the power intensity vector)

$$\widetilde{\mathbf{I}}^+ \;\equiv\; \mathbf{e}\,\widetilde{I}^+ = P^+\mathbf{v}^+ = \mathbf{e}P^+v^+, \tag{65}$$
$$\begin{aligned}\widetilde{\mathbf{I}}^- \;&\equiv\; -\mathbf{e}\,\widetilde{I}^- = P^-\mathbf{v}^- = \mathbf{e}P^-v^-\\ &=\; \mathbf{e}\mathsf{R}_v\mathsf{R}_pP^+v^+ = -\left(R_v+r_vv^+\right)\left(R_v+r_pP^+\right)\widetilde{\mathbf{I}}^+,\end{aligned} \tag{66}$$
$$\begin{aligned}\widetilde{\mathbf{I}}_2 \;&\equiv\; \mathbf{e}\,\widetilde{I}_2 = P_2\mathbf{v}_2 = \mathbf{e}P_2v_2\\ &=\; \mathbf{e}\mathsf{T}_v\mathsf{T}_pP^+v^+ = \left(T_v+r_vv^+\right)\left(T_v\frac{z_{02}}{z_{01}}-r_pP^+\right)\widetilde{\mathbf{I}}^+.\end{aligned} \tag{67}$$

The quantities defined above satisfy Eq. (64) under the following condition

$$\left(v^+r_v-P^+r_p\right)=0+O(q^3). \tag{68}$$

We shall return to this equation later. Now, we would like to notice that from the continuity equations (38), (39) we obtain a general form of the conservation law of the energy current density vector (64), namely

$$\widetilde{\mathbf{I}}_1=\widetilde{\mathbf{I}}_2 \;\Leftrightarrow\; \widetilde{I}_1=\widetilde{I}_2\,, \tag{69}$$

$$P^{+}v^{+} + P^{-}v^{-} + (v^{+}P^{-} + P^{+}v^{-}) = P_2 v_2 \,. \tag{70}$$

This means that the decomposition $\widetilde{\mathbf{I}}_1 = \widetilde{\mathbf{I}}^{+} + \widetilde{\mathbf{I}}^{-}$, although it seems to be natural, is nevertheless true under the condition

$$(v^{+}P^{-} + P^{+}v^{-}) = 0. \tag{71}$$

If we suppose (71), then the condition (68) is satisfied. This is easy to see after the substitution of P^{-} and v^{-} with the right-hand sides of (59) and (61). But (71) has a more primary meaning; which will be demonstrated below.

Replacing P^{-} and v^{-} in (71) by the right-hand sides of (59) and (61), and using the impedance relations (35) or (32), we obtain

$$(v^{+}\mathsf{R}_v v^{+} + v^{+}\mathsf{R}_p v^{+}) - (\partial_x \Phi^{+}\mathsf{R}_v v^{+} + v^{+}\mathsf{R}_p \partial_x \Phi^{+}) = 0 \tag{72}$$

or

$$(P^{+}\mathsf{R}_v P^{+} + P^{+}\mathsf{R}_p P^{+}) + z_{01}(\partial_x \Phi^{+}\mathsf{R}_p P^{+} + P^{+}\mathsf{R}_v \partial_x \Phi^{+}) = 0. \tag{73}$$

We suppose that

$$\mathsf{R}_p = -\mathsf{R}_v \,. \tag{74}$$

From (72), (73) we have the following "commutation" relations

$$\partial_x \Phi^{+} \cdot \left(\mathsf{R}_{v \text{ or } p} \begin{pmatrix} v^{+} \\ P^{+} \end{pmatrix} \right) - \begin{pmatrix} v^{+} \\ P^{+} \end{pmatrix} \cdot \left(\mathsf{R}_{v \text{ or } p} \partial_x \Phi^{+} \right) = 0. \tag{75}$$

On the other hand, replacing P^{-}, P^{+} (or v^{-}, v^{+}) in (71) by the impedance relations (33) and (34), using the relation (59) (or (61)), and applying (75), we obtain

$$\partial_x \Phi^{-} \;=\; \mathsf{R}_v \partial_x \Phi^{+}, \tag{76}$$

$$\partial_x \Phi^{-} \;=\; -\mathsf{R}_p \partial_x \Phi^{+}. \tag{77}$$

Of course, these relations can be received also from (72), (73), particularly if we have (68). Equations (74), (75) permit to perform a separation process on the "couple equation" (42) or (44). After substitution the right hand sides of (61) and (77) into (42), we have

$$\partial_x \Phi^{+} = \mathsf{T}_p^{-1} \left(\frac{1}{z_{01}} \overline{\mathcal{A}}_1 \mathsf{T}_p P^{+} - \frac{q_1 c_{10}}{2 z_{01}^2} \left(\mathsf{T}_p P^{+} \right)^2 \right) + O\big((q+\alpha)^2\big) \,. \tag{78}$$

However, consequently in our order of approximation, we can replace T_p by T_p in (78)

$$\mathsf{T}_p \;=\; (1 + \mathsf{R}_p) = T_p + O(\alpha + q) = 1 + R_p + O(\alpha + q), \tag{79}$$

$$\partial_x \Phi^{+} \;=\; \frac{1}{z_{01}} \overline{\mathcal{A}}_1 P^{+} - \frac{q_1 c_{10}}{2 z_{01}^2} T_p \left(P^{+} \right)^2 + O\big((q+\alpha)^2\big)$$

and in the described approximation ($\mathcal{A}_m = 0$)

$$\partial_x \Phi^{+} = -\frac{q_1 c_{10}}{2 z_{01}^2} T_p \left(P^{+} \right)^2 + O\big((q+\alpha)^2\big) \,. \tag{80}$$

Nevertheless, if we have the velocity reflection operator $\mathsf{R}_v[\,\cdot : v^+]$ and want to determine the pressure reflection operator R_p as a function of P^+, using (32) and (74), we obtain

$$\mathsf{R}_p\left[\,\cdot\,; P^+\right] = -\mathsf{R}_v\left[\,\cdot\,; \frac{P^+}{z_{01}} + \partial_x \Phi^+\right], \tag{81}$$

where $\partial_x \Phi^+$ is given by (78). In our case this relation reduces to (68). Applying (80) we obtain

$$\left[\left(1 - \frac{q_1 c_{10}}{2z_{01}} T_p P^+\right) \frac{r_v}{z_{01}} - r_p\right] P^+ = 0 + O(q^3), \qquad T_p = \frac{2z_{02}}{z_{01} + z_{02}}\,. \tag{82}$$

In our range of the approximation of $\mathsf{R}_v[\,\cdot : v^+]$, r_v is a constant with respect to v^+ and is of $O(q)$. To be consistent, we must neglect the second term in the internal parenthesis, which leads to

$$r_p = (r_v/z_{01}) + O(q^2), \tag{83}$$

and

$$\begin{aligned} P^-(t) &= -\left[\frac{z_{01} - z_{02}}{z_{01} + z_{02}} \right. \\ &\quad \left. + \frac{2z_{02}q}{(z_{01} + z_{02})^3} \left(\frac{z_{02}}{c_{10}}\beta_1 - \frac{z_{01}}{c_{20}}\beta_2\right) P^+(t)\right] P^+(t) + O(q^2), \end{aligned} \tag{84}$$

$$\begin{aligned} P_2(t) &= \left[\frac{2z_{02}}{z_{01} + z_{02}} \right. \\ &\quad \left. - \frac{2z_{02}q}{(z_{01} + z_{02})^3} \left(\frac{z_{02}}{c_{10}}\beta_1 - \frac{z_{01}}{c_{20}}\beta_2\right) P^+(t)\right] P^+(t) + O(q^2). \end{aligned} \tag{85}$$

In the Fourier frequency domain

$$\widehat{\mathsf{R}}_v = \left[\frac{z_{01} - z_{02}}{z_{01} + z_{02}} + \frac{2z_{02}z_{01}q}{(z_{01} + z_{02})^3} \left(\frac{z_{02}}{c_{10}}\beta_1 - \frac{z_{01}}{c_{20}}\beta_2\right) \hat{v}^+(\omega)\otimes\right] + O(q^2), \tag{86}$$

$$\widehat{\mathsf{R}}_p = -\left[\frac{z_{01} - z_{02}}{z_{01} + z_{02}} + \frac{2z_{02}q}{(z_{01} + z_{02})^3} \left(\frac{z_{02}}{c_{10}}\beta_1 - \frac{z_{01}}{c_{20}}\beta_2\right) \widehat{P}^+(\omega)\otimes\right] + O(q^2). \tag{87}$$

The following term

$$\hat{r}_A \equiv \frac{2z_{01}z_{02}}{(z_{01} + z_{02})^2} \frac{(a^2(\omega) - a^1(\omega))}{-i\omega}\,, \tag{88}$$

should be added to the right-hand side of (86) in order to take into account the absorption.

6. Separation of propagation equations

The operators of reflection and transmission were derived in the case of a boundary between two different adjacent media. This is a basic problem, however, its solution does not close the description of the reflection and transmission phenomenon in the case of nonlinear propagation (contrary to the linear one). Figure 2 shows a time-space diagram corresponding to Fig. 1, which makes it possible to explain exactly the phenomenon and the notion of the nonlinear reflection and transmission.

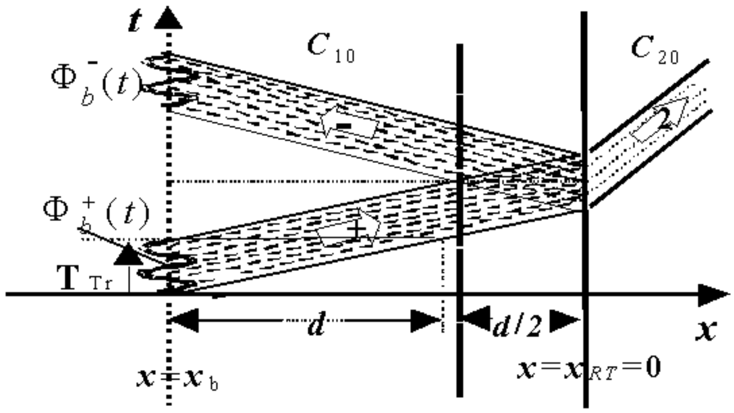

Fig. 2. Time-space diagram of reflection and transmission. "+", "−", "2" – time-space ribbons of incident, reflected and transmitted beams, respectively; d is the spatial length of the pulse; $d/2$ is the maximum thickness of the interaction boundary layer near to interface.

According to the previous assumptions (Sec. 3), the point x_b is the point of the boundary condition (excitement point) for the incident pulse or the point where the pulse leaves time traces with the duration of T_{Tr}. So the length of the pulse equals $d = T_{\mathrm{Tr}} c_{10}$. The ribbons signs by "+", "−" and "2" represented the time-space traces of the incident, reflected and transmitted pulses, respectively. The triangular area of the time-space is the area of interaction of the reflected part of the pulse with a part of the incident pulse. The "couple equation" (27) or (42), which were utilized previously only on the boundary $x = x_{RT} = 0$, is now written below as to be valid in the whole area $x \le x_{RT} = 0$,

$$\left(\partial_x \Phi^+ - \partial_x \Phi^-\right) = \frac{1}{z_{01}} \overline{\mathcal{A}}_1 \left(P^+ + P^-\right) - \frac{q_1 c_{10}}{2 z_{01}^2} \left(P^+ + P^-\right)^2 + O\left((q+\alpha)^2\right), \qquad (89)$$

This interaction is described by the term proportional to P^+P^-. In the linear description, only the linear superposition of disturbances arises in this area. A specific feedback occurs in the nonlinear case, where the reflected part of the pulse can cause changes in the amplitude of the incident pulse (and changes in the amplitudes of the component disturbances in the boundary conditions at $x_{RT} = 0$). Taking into account this point of view, one can say that the whole interaction boundary layer witha thickness changing in time (the maximum thickness equals $d/2$) reflects and transmits the incident pulse. It arises the question how to determine Φ^+, especially on the interface and Φ^- for $x \le x_{RT} = 0$ – and which equations are responsible for the evolution of these fields? Let us notice, that the determination of the reflection and transmission operators for the velocity does not cause any additional conditions for (27) or its boundary (limiting) values expressed by (42) and (44). This means that the partition of (26), (27) or (89) into equations describing the evolutions of Φ^+ and Φ^- is partially arbitrary. However, taking into account the physical interpretation (see end of Sec. 3) of the nonlinear terms in (89), in the asymptotic area outside of the interaction layer, where $qP^+P^- = 0 + O(q\alpha)$, every of the equations should be the Burger's equation or should give the Burger's equation after application of $-g_{0m}\partial_t$ to both the sides of them.

Let us consider the following system of equations,

$$\begin{aligned} \partial_x\Phi^+ &= \frac{1}{z_{01}}\overline{\mathcal{A}}_1 P^+ - \frac{q_1 c_{10}}{2z_{01}^2}\left(P^+ + \varepsilon P^-\right)P^+ + O\big((q+\alpha)^2\big), \\ P^+ &= -g_{01}\partial_t\Phi^+ = -g_{01}\partial_{\tau^+}\Phi^+, \end{aligned} \tag{90}$$

$$\begin{aligned} -\partial_x\Phi^- &= \frac{1}{z_{01}}\overline{\mathcal{A}}_1 P^- - \frac{q_1 c_{10}}{2z_{01}^2}\left(P^- + (2-\varepsilon)P^+\right)P^- + O\big((q+\alpha)^2\big), \\ P^- &= -g_{01}\partial_t\Phi^- = -g_{01}\partial_{\tau^-}\Phi^-, \end{aligned} \tag{91}$$

where ε is an arbitrary parameter (however, we suppose that the limitation for ε is $\varepsilon q \le O(q)$). The sum of the sides of (90), (91) gives (89). Applying $-g_{01}\partial_t$ to both the sides of (90) and (91), we obtain in the asymptotic area the Burger's equations for P^+ and P^-, respectively. That means that Eqs. (90), (91) fulfil the above assumption. Generally speaking, if Φ^+ and Φ^- are solutions of the above system of equations, then they also fulfil Eqs. (89) and (13). This will be discussed more extensively latter. Because $2P^+P^- = \varepsilon\, P^+P^- + (2-\varepsilon)P^+P^-$, the parameter ε, introduced by us in the description of propagation given by (90) and (91), shows the partition of interaction between P^+ and P^- (Φ^+ and Φ^-). Of course, the partition of (89) expressed by Eqs. (90), (91) is a result of the same partition of (13) and of applying the conditions given in Secs. 2, 3.

Expressing the velocities in the condition of (39) by means of the corresponding pressures given by the impedance relations (32), (34), and applying the interface values of $\partial_x\Phi^-$, $\partial_x\Phi^+$ and $\partial_x\Phi_2$ resulting from Eqs. (90), (91) and (43), we obtain for R_p^ε the formulas

$$\mathsf{R}_p^\varepsilon\left[\{m\};P^+\right] \equiv \frac{1}{2\mathsf{w}_p \circ \mathsf{R}_{0p}^\varepsilon P^+}\left[1 - \sqrt{1 - 4\mathsf{w}_p \circ \mathsf{R}_{0p}^\varepsilon P^+}\right]\mathsf{R}_{0p}^\varepsilon, \tag{92}$$

$$\mathsf{R}_{0p}^\varepsilon\left[\{m\};P^+\right] \equiv \frac{\mathsf{z}_{p2} - \mathsf{z}_{p1} - \dfrac{1}{2}q z_{01} z_{02}\left(\left(\beta_1/c_{10}z_{01}^2\right) - \left(\beta_2/c_{20}z_{02}^2\right)\right)P^+}{\mathsf{z}_{p1} + \mathsf{z}_{p2} - w_\varepsilon P^+}, \tag{93}$$

$$\mathsf{w}_p\left[\{m\};P^+\right] \equiv \frac{\frac{1}{2}qz_{01}z_{02}\left(\left(\beta_1/c_{10}z_{01}^2\right)+\left(\beta_2/c_{20}z_{02}^2\right)\right)\cdot}{\mathsf{z}_{p1}+\mathsf{z}_{p2}-w_\varepsilon\cdot P^+}, \tag{94}$$

$$\begin{aligned} w_\varepsilon &\equiv qz_{01}z_{02}\left((1-\varepsilon)\left(\beta_1/c_{10}z_{01}^2\right)+\left(\beta_2/c_{20}z_{02}^2\right)\right), \\ \mathsf{z}_{p1} &\equiv z_{01}\left(1+\mathcal{A}_2\right), \\ \mathsf{z}_{p2} &\equiv z_{02}\left(1+\mathcal{A}_1\right). \end{aligned} \tag{95}$$

With the hitherto existing accuracy (under the assumption $\mathcal{A}_1 = 0 = \mathcal{A}_2$), we obtain

$$\mathsf{R}_p^\varepsilon = -\left\{\frac{z_{01}-z_{02}}{z_{01}+z_{02}} + \frac{q2z_{02}z_{01}}{\left(z_{01}+z_{02}\right)^3}\left[\left(\frac{z_{02}}{z_{01}}+\frac{\varepsilon-1}{2}\left(\left(\frac{z_{02}}{z_{01}}\right)^2-1\right)\right)\frac{\beta_1}{c_{10}} - \frac{\beta_2}{c_{20}}\right]P^+(t)\right\} + O(q^2). \tag{96}$$

By the term “particular description” we will denote here the situation in which the value of ε is fixed. From the above is evident that the number of “particular descriptions” can be arbitrarily large. It may also cause the justified impression that different values of the transmitted, reflected and incident fields *in the layer* for the same disturbance incident *on the layer* correspond to every particular description. Especially the change of ε causes changes of the boundary condition expressed by pressure for Φ_2 (exactly for P_2) for the same disturbance incident on the layer. This means that the “descriptions” are not synonymous. The unsolved case of the equivalence of the descriptions of pressures for various ε means that the problem is not closed and means also a lack of the energy current description.

The problem is not trivial because even the assumption of the existence and uniqueness of the solutions of the boundary problems (at x_b) for all the equations presented (especially if depending on ε) does not solve automatically the problem of equivalence of the “particular descriptions”.

We remind here that we have three “modes” and only one of them is “fixed” by means of the boundary condition at $x = x_b$ in the asymptotic area.

One should show that the value of $P^+ + P^-$ does not depend on ε. This means, especially for $x = x_{RT}$, that there is no additional nonlinear effect depending on ε, i.e. such an effect that the boundary condition $P^+ + P^- = P_2$ is always fulfilled, however on other levels of amplitudes depending on ε. This means that P_2 does not depend on ε either, in spite of the fact that the boundary relation P^+ and P^- expressed by R_p^ε depends evidently on ε.

Assuming the existence and uniqueness of the solutions of the boundary problems for the differential equations, especially for the Eqs. (13), (90) and (91) applied in this paper; it can be shown that:

If for an arbitrary given ε Φ^{+} and Φ^{-} are uniquely solutions of the Eqs. (90), (91) for $x_b \le x \le x_{RT}$ and Φ_2 for $x \ge x_{RT}$, the unique solution of the equation

$$\begin{aligned} \partial_x \Phi_2 &= \frac{1}{z_{02}} \overline{\mathcal{A}}_2 P_2 - \frac{q_2 c_{20}}{2 z_{02}^2} (P_2)^2 + O\big((q+\alpha)^2\big) , \\ P_2 &= -g_{02} \partial_t \Phi_2 = -g_{02} \partial_{\tau_2} \Phi_2 , \end{aligned} \tag{97}$$

where Φ^{+} fulfills the boundary condition for $x = x_b$

$$\Phi^{+}\big|_{x_b} = \Phi_b^{+}(t), \tag{98}$$

Φ^{-} and Φ_2 fulfill the boundary condition at $x = x_{RT}$

$$P^{-} = \mathsf{R}_p^{\varepsilon} P^{+} = \mathsf{R}_p^{\varepsilon} P^{+}(\tau^{+}(x_{RT}, t), x_{RT}), \tag{99}$$

$$P_2 = (P^{+} + P^{-}) = \mathsf{T}_p^{\varepsilon} P^{+}, \tag{100}$$

then:

1. The functions Φ^{+}, Φ^{-} and Φ_2 fulfill (89) and (97) for $x_b \le x \le x_{RT}$ and $x \ge x_{RT}$, respectively, so they fulfill also (24) and (26). This means that

$$\Phi_1(x,t) = \Phi^{+}(\tau^{+}(x,t), x) + \Phi^{-}(\tau^{-}(x,t), -x), \tag{101}$$

is the solution of (13) (with the accuracy $O((q+\alpha)^2)$ having at $x = x_b$ the values

$$\begin{aligned} \Phi_1(x_b, t) &= \Phi^{+}(\tau^{+}(x_b,t), x_b) + \Phi^{-}(\tau^{-}(x_b,t), -x_b) \\ &= \Phi_b^{+}(t) + \Phi_b^{-}(t) \qquad \text{for} \quad 0 \le t < \infty, \end{aligned} \tag{102}$$

2. For $0 \le t < \infty$, $\Phi_1 = \Phi^{+} + \Phi^{-}$ Φ_2, and also $P_1 = P^{+} + P^{-}$ and P_2 do not depend on ε.

$\Phi_b^{-}(t)$ is the non evident (being searched) component of the boundary condition. In our problem we consider it as a "time trace" which leaves the reflected disturbance in $x = x_b$. This component must be formally included in the correctly formulated mathematical description of the boundary problem if it is analyzed for $0 \le t < \infty$.

Apart from above assumptions the thesis **2.** results also from the following equations

$$(P^{+} + P^{-}) = z_{01} \left(v^{+} - v^{-} - \overline{\mathcal{A}}_1 \left(v^{+} - v^{-} \right) + \frac{q_1 c_{10}}{2} \left(v^{+} - v^{-} \right)^2 \right), \tag{103}$$

which were used previously in Sec. 4 to determine R_v. The right hand side of Eq. (103) does not depend on ε. P_ε^{+}, and P_ε^{-} depend on the ε (here we evidently denote this fact by subscript ε), however $P_1(x,t) = P_\varepsilon^{+} + P_\varepsilon^{-}$ and $P_2(x,t)$ do not depend on ε. If P_ε^{+} and P_ε^{-} are solutions of the Eqs. (90), (91) for the same boundary conditions at $x = x_b$ but for different values $\varepsilon 1 \ne \varepsilon 2$, then $P_{\varepsilon 1}^{+} + P_{\varepsilon 1}^{-} = P_1 = P_{\varepsilon 2}^{+} + P_{\varepsilon 2}^{-}$. Especially at $x = x_{RT}$

$$P_1 = \left(1 + \mathsf{R}_p^{\varepsilon 1}\right) P_{\varepsilon 1}^{+} = \mathsf{T}_p^{\varepsilon 1} P_{\varepsilon 1}^{+} = \left(1 + \mathsf{R}_p^{\varepsilon 2}\right) P_{\varepsilon 2}^{+} = \mathsf{T}_p^{\varepsilon 2} P_{\varepsilon 2}^{+} = P_2 \,. \tag{104}$$

For $\varepsilon = 1$ we obtain (87) as a particular case of the asymptotic formula (96). In this case the system of equations (90) and (91) shows the symmetry in the description of the interaction in relation to other possibilities, i.e. when $\varepsilon \neq 1$. On the surface $x = x_{RT}$, the energy current density vector $\widetilde{\mathbf{I}}_1 = P_1\,\mathbf{v}_1$ takes the form (64) and the Eq. (90) gives (78) with the hitherto existing accuracy. The case $\varepsilon = 0$ will be discussed in the next section. For $\varepsilon = \varepsilon_L \equiv 1 + T_p((\beta_{21}/c_{210}z_{021}) - 1)/(z_{021} - 1)$, where $z_{021} \equiv z_{02}/z_{01}$, $c_{210} \equiv c_{20}/c_{10}$, $\beta_{21} \equiv \beta_2/\beta_1$, the formulas (92) and (96) are reduced to $\mathsf{R}_p^{\varepsilon_L} = R_p$.

Hence, one can conclude that choosing of $\varepsilon = \varepsilon_L$ the "linearization" of the boundary relations between P^-, P_2 and P^+ can be performed. Nevertheless, from (104) we have $\mathsf{T}_p^\varepsilon P_\varepsilon^+ = T_p P_{\varepsilon_L}^+$ for every ε. In this case the local and nonlinear description of the influence of the boundary on the transmission and reflection for pressure was "translated" into the interior of the layer. This is an interesting example showing the function of the interaction boundary layer in the phenomenon of reflection and transmission and in its description.

7. Discussion and conclusions

The determined operators of reflection and transmission for pressure and velocity preserve the asymptotic properties of the linear operators for $(z_{01}; z_{02}) \to (0 \text{ or } \infty)$, except the situations $z_{01} \to 0$; $z_{02} \to \infty$, when the limit of the R_p^ε is a function of ε (however $\mathsf{R}_p^{\varepsilon=1} \to 1$). It follows from (104) that the transmission of energy to the medium $m = 2$, $\widetilde{\mathbf{I}}_2 \equiv P_2\mathbf{v}_2 = \mathbf{e}\mathsf{T}_v v^+ \mathsf{T}_p^\varepsilon P^+$ does not depend on ε. Additionally, the transmission of energy to the medium $m = 2$ vanishes in all the asymptotic cases because either $\mathsf{T}_v \to 0$ or $\mathsf{T}_p^\varepsilon \to 0$. The transmission of energy to the medium $m = 2$ is complete under the conditions $\mathsf{R}_0 = 0$, $\mathsf{R}_{0p}^\varepsilon = 0$ (see (51), (53) and (92), (93)). These conditions also do not depend on ε and are fulfilled if the relations $z_{01} = z_{02}$, $\mathcal{A}_1 = \mathcal{A}_2$, $(\beta_1/c_{10}) = (\beta_2/c_{20})$ are kept.

Let us notice that the lack of differences between the nonlinear parameters in both the media $(\beta_1 = \beta_2)$ is not a sufficient condition for the vanishing of the nonlinear component of the reflected component of the wave with respect to the incident wave. This condition can be written in the form

$$z_{02}c_{20}\beta_1 = z_{01}c_{10}\beta_2\,. \tag{105}$$

However, then $\mathsf{R}_v = -\mathsf{R}_p^{\varepsilon=1}$, where

$$\mathsf{R}_v = \frac{z_{01} - z_{02}}{z_{01} + z_{02}} = \frac{\beta_1 c_{20} - \beta_2 c_{10}}{\beta_1 c_{20} + \beta_2 c_{10}}\,. \tag{106}$$

In the case $\beta_1 \neq \beta_2$ this means that, in spite of a linear dependence between the reflected and incident disturbances, the phenomenon of the interaction with the interface preserves its nonlinear character as before.

Hence, in the case of nonlinear propagation it results that the basic nonlinear effect of reflection and transmission depends in the same degree on all the parameters of the media in the equilibrium state (for small signals) and on the parameters characteristic for the nonlinear properties of the media (nonlinearity of the state equations). However, it is possible to assume that the pure nonlinear reflection and transmission occurs only when $\beta_1 \neq \beta_2$. In such a case we obtain

$$P^-(t) = -\frac{q}{4z_{01}c_{10}}(\beta_1 - \beta_2)P^+(t)^2 + O(q^2), \tag{107}$$

$$v^-(t) = \frac{q}{4c_{10}}(\beta_1 - \beta_2)v^+(t)^2 + O(q^2), \tag{108}$$

$$\widehat{P}^-(\omega) = -\frac{q}{4z_{01}c_{10}}(\beta_1 - \beta_2)\widehat{P}^+(\omega) \otimes \widehat{P}^+(\omega) + O(q^2), \tag{109}$$

$$\hat{v}^-(\omega) = \frac{q}{4c_{10}}(\beta_1 - \beta_2)\hat{v}^+(\omega) \otimes \hat{v}^+(\omega) + O(q^2). \tag{110}$$

In general (taking into account the remark according to (106)) the reflected (or transmitted) disturbance can be written in the form

$$v^- = v_L^- + v_{NL}^-, \qquad P^- = P_L^- + P_{NL}^-, \tag{111}$$

where

$$v_L^-(t) \equiv R_v v^+(t), \qquad P_L^-(t) \equiv R_p P^+(t), \tag{112}$$

$$v_{NL}^-(t) \equiv r_v v^+(t) \cdot v^+(t), \qquad P_{NL}^-(t) \equiv -r_p P^+(t) \cdot P^+(t), \tag{113}$$

and the index L denotes to the linear component with respect to the disturbance but not to the linear one with respect to the description of the phenomenon. It follows that in the Fourier frequency domain

$$\hat{v}_L^-(\omega) \equiv R_v \hat{v}^+(\omega), \qquad \widehat{P}_L^-(\omega) \equiv R_p \widehat{P}^+(\omega), \tag{114}$$

$$\hat{v}_{NL}^-(\omega) \equiv r_v \hat{v}^+(\omega) \otimes \hat{v}^+(\omega), \qquad \widehat{P}_{NL}^-(\omega) \equiv -r_p \widehat{P}^+(\omega) \otimes \widehat{P}^+(\omega). \tag{115}$$

From the obtained results, especially from (115), it can be concluded that the reflection and transmission for nonlinear propagation is not a local phenomenon in the frequency domain. The reflection and transmission of every of the Fourier components of the incident wave depends on all the remaining components as it follows from the properties of the auto convolution. In the nonlinear description of the interaction with the boundary surface the reflection and transmission of the single Fourier component is not independent of the remaining spectral components.

When the incident wave is generated by a pulse transmitter with the carrier frequency of ω_{ca} in the neighborhood of the interface, then the even for the relatively high values of q, the spectrum of the incident wave $(v^+(\omega), P^+(\omega))$ will be a distinctly visible a single spectral line concentrated around ω_{ca} as before around ω_{ca}. As it results from (114) and (115) in the spectrum of the reflected (or transmitted) wave, beside the spectral line with the same central frequency there will arise an additional spectral line distinctly separated (for the adequately pulse length) that corresponds to (v^-_{NL}, P^-_{NL}) and is concentrated around $2\omega_{ca}$. However, this is rather an exceptional situation. In general, the spectra v^+, P^+ of the disturbance reaching the reflecting interface during nonlinear propagation are already complex. With the exception of hypothetical situations and in the situation described above, the spectra (v^-_{NL}, P^-_{NL}) carry much lower energy than (v^-_L, P^-_L) (even if $z_{01} \approx z_{02}$) and are practically overlapped by them (in the lower spectral range). Hence it results a practical significance of the values (v^-_{NL}, P^-_{NL}) in the analysis of different properties of the both the media (for example the determination of β for one of the medium when the other ones are known) depends on the application of special techniques of excitation and detection of the reflected or transmitted waves. In this context it should be stessed that the change of the sign of v^+, P^+ does not cause a change of the sign in (v^-_{NL}, P^-_{NL}). Examples of quantitative comparisons of (v^-_{NL}, P^-_{NL}) with (v^-_L, P^-_L), which are decisive for the standard detection technique used in ultrasonography are given in the Appendix A4.

An interesting and curious property of the nonlinear reflection and transmission is the fact that at the point of discontinuity of the media parameters, i.e. on zero of the dimensional manifold, there arises the effect of finite nonlinear change of pressure (velocity). It is of the same order of magnitude as the nonlinear change of pressure resulting from nonlinear propagation on the finite distance of δx, that means on one-dimensionally manifold with non zero measure, in the first or second component of the medium. Details of the reasoning, which makes possible the estimation of the effect, are here omitted

$$\delta x(\omega) = \text{const}(z_{021}, \beta_{21}) \frac{c_{10}}{\omega} \,. \tag{116}$$

Equation (116) has a qualitative meaning characterizing the comparison effect. For the derivation of the reflection and transmission operators no additional assumptions were necessary apart from those formulated at the beginning of this paper. To obtain analogical operators for pressure the additional partition of (26) and (27) was necessary. However, this did not allow us to close the description of the phenomenon on the base of the hitherto existing assumptions. The assumption of the partition of the energy current, though characteristic due to the symmetry of interaction, is a consequence of one of the possible particular descriptions of interaction in the layer. In the general case and because of the evidently occurring additional component in R^ε_p that depends on the description of the interaction, the additional discussion in Sec. 6 was necessary. Hence it follows:

1. The relative significance of the *individual components,* which form the representation of the nonlinear disturbance, in contrast to the full representation (which is equivalent to exact solution of the problem). In the linear case such a differentiation has no meaning. In our case, there exist a comprehensive set of functions $\{P^+_\varepsilon, P^-_\varepsilon\}$ "numbered" by ε and realizing the same P_1 for every ε. Because ε can be changed arbitrarily, then the functions P_ε^+ and P_ε^- separately considered have no physical meaning in the area of interaction where $P_\varepsilon^+ P_\varepsilon^- \neq 0$. On the contrary to this, $P_1 = P_\varepsilon^+ + P_\varepsilon^-$ is a measurable quantity equivalent to the exact solution.

2. The independence of $\Phi_1 = \Phi^+ + \Phi^-$ and Φ_2, (or $P_1 = P^+ + P^-$ and P_2) on ε makes it possible to choose such a particular description for which the solution of Eqs. (90), (91), (97) (and also (13)) is the simplest one (even from the numerical point of view). For example, assuming a medium with a classical viscosity and $\varepsilon = 0$ we conclude, that (90) and (97) are Burgers equations (integrated over the time – see the remarks before and after (90), (91)). Boundary problems for those equations are exactly integrable. Thus the factor $2P^+$ in the term describing interaction in Eq. (91) is defined for every t. These equations can be written as follows

$$\partial_\xi \Phi^-(\tau^-, \xi) + \frac{1}{z_{01}} \alpha_2^1 \partial_t P^-(\tau^-, \xi) + \frac{q_1 c_{10}}{2z_{01}^2} \Big[P^-(\tau^-, \xi)^2 + 2P^+\left(\tau^- + \frac{2\xi}{c_{10}}, \xi\right) P^-(\tau^-, \xi) \Big] = 0 + O((q+\alpha)^2). \tag{117}$$

The following substitutions were applied here: $-x = \xi$, and $\tau^+ = t - x/c_{10} = \tau^- + 2\varsigma/c_{10}$. The same transformation (Cola–Hopf, $\Phi = \text{const} \cdot \ln(\Psi)$), which transforms (90) and (97) into linear equations, transforms (91) or (117) into quasi-linear equations and, in the asymptotic area, into linear ones. The analysis in Sec. 6 is of additional advantage and, in the case of a classical absorption of both the components, an exact solution for $\Phi_2(P_2)$ can be obtained.

3. We would like to stress that the parameter of the description (scaling) of nonlinear interaction can be the arbitrary function $\varepsilon(x,t)$ (however, $\varepsilon(x,t)\, q \le O(q)$). This allows us to see the problem of the choice of the description of interaction as the choice of a special "functional system" of coordinates. Nevertheless, every choice maintains the value $\Phi_1, \Phi_2\ (P_1, P_2)$.

It may by interesting that the author has obtained the same form of the reflection and transmission operators (for v^+ and $-g_{01}\partial_t \Phi^+$) presented in this paper on the base of the Kuznetsov equation using a definition of acoustical pressure presented in formula (A10) in Appendix A1 (in the Kuznetsov approximation $-g_{01}(\partial_t \Phi^+ + \partial_t \Phi^-) \neq P_1$).

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Appendix

A1

The pressure tensor $\mathbf{\Pi}$ has the form [8]

$$\mathbf{\Pi} = P_{th}(g,s)\mathbf{1} + 2A^{Ls}, \tag{A1}$$

where P_{th} is the thermodynamic pressure; g, s are density and entropy, respectively; $\mathbf{1}$ is the unit tensor; $\mathsf{A}^{Ls} = \mathsf{a}\mathbf{1} + \mathbf{A}^S = \left(\mathsf{a} + \mathrm{Tr}(\mathbf{A}^S)\right)\mathbf{1} + \left(\mathbf{A}^S - \mathrm{Tr}(\mathbf{A}^S)\mathbf{1}\right)$ is the viscosity stress tensor. A^{Ls} is linear with respect to the components v_j of $\mathbf{v}$ and symmetrical $\left[\mathbf{A}^S\right]_{l,j} = \mathcal{A}_l^S v_j = \mathcal{A}_j^S v_l$, $j,\ l = 1,2,3$. Calculating its divergence and taking into account that $\nabla\cdot\left(\mathbf{A}^S - \mathrm{Tr}(\mathbf{A}^S)\mathbf{1}\right) = \mathbf{0}$ and $\mathbf{v} \equiv \nabla\Phi$, we obtain the total force per unit volume of the medium

$$\nabla\cdot\mathbf{\Pi} \;=\; \nabla\widetilde{P} = \nabla\left(P_{th}(g,s) + 2\mathcal{A}_{Ls}\Phi\right), \tag{A2}$$

$$\mathcal{A}_{Ls}\Phi \;=\; \mathsf{a} + \mathrm{Tr}(\mathbf{A}^s). \tag{A3}$$

For classic the Navier–Stokes model: $\mathsf{a} = -\left((\eta_b/2) - \eta_s/3\right)\nabla\cdot\mathbf{v}$, $\mathcal{A}_l^S = -\eta_s\partial/\partial x_l$, $\mathcal{A}_{Ls} = -\alpha_{2Ls}\Delta$, $\alpha_{2Ls} \equiv (4\eta_s/3 + \eta_b)/2$, where η_s, η_b are the shear and bulk viscosity, respectively. For a nearly adiabatic conversion of the medium and in the first order with respect to the entropy variations $P_{th}(g,s) = P^* + 2\mathcal{A}_{th}\Phi$ [9], where P^* describes the pressure in the adiabatic conversion; $\mathcal{A}_{th} \equiv -\alpha_{2th}\Delta$. In such a case, the pressure $\widetilde{P}$ can be written as

$$\widetilde{P}(g,\Phi) = P^*(g) + 2\mathcal{A}\Phi, \qquad \mathcal{A} = \mathcal{A}_{th} + \mathcal{A}_{Ls}\,. \tag{A4}$$

For classical viscous media $A = -(\alpha_{2th} + \alpha_{2Ls})\Delta = -\alpha_2\Delta$. Nevertheless, the full operator

$$2\mathcal{A}\Phi = 2(\mathcal{A}_{th} + \mathcal{A}_{Ls})\Phi = \widetilde{P}(g,s;\Phi) - P^*(g) + O(s^2) \tag{A5}$$

may be reconstructed on the base of measurements of the small signal coefficient of absorption $a(\omega)$ [7]. After substituting $\widetilde{P}(g,\Phi) = P^*(g) + 2\mathcal{A}\Phi$ into the momentum equation, where for m-th medium

$$P^*(g_m) = \frac{g_{0m}c_{m0}^2}{q\gamma_m}\left(\frac{g_m}{g_{0m}}\right)^{\gamma_m} \tag{A6}$$

or for the empirical equations of state, $\gamma_m \equiv 1 + (B/A)_m$

$$P^*(g_m) = P_{m0} + \frac{g_{0m}c_{m0}^2}{q}\left[\left(\frac{g_m}{g_{0m}} - 1\right) + \frac{\gamma_m - 1}{2}\left(\frac{g_m}{g_{0m}} - 1\right)^2 + O(q^3)\right], \tag{A7}$$

we have in both cases

$$\begin{aligned} g_m(\mathbf{x},t) \;=\; g_{0m}\Bigg[1 - \frac{q}{c_{m0}^2}\Bigg(\partial_t\Phi + \frac{q}{2}\,(\nabla\Phi)^2 \\ + \frac{q(\gamma_m - 2)}{2c_{m0}^2}\,(\partial_t\Phi)^2 + 2\mathcal{A}_m\Phi\Bigg)\Bigg] + O(q\alpha(q+\alpha)), \end{aligned} \tag{A8}$$

$$c_m^2(\mathbf{x},t) = c_{m0}^2\Bigg[1 - \frac{q(\gamma_m-1)}{c_{m0}^2}\Bigg(\partial_t\Phi + \frac{q}{2}(\nabla\Phi)^2 + \frac{q(\gamma_m-2)}{2c_{m0}^2}(\partial_t\Phi)^2 + 2\mathcal{A}_m\Phi\Bigg)\Bigg] + O(q\alpha(q+\alpha)). \qquad \text{(A9)}$$

From the mass continuity equation and formula (A8) we can obtain (in the first order with respect to q and α) the Kuznetsov equations [10] and the (13) [7].

The disturbance of the total pressure $\widetilde{P}$ from the equilibrium pressure P_{m0} takes the form

$$\begin{aligned} P_m(\mathbf{x},t) &= \mathsf{P}_m[\Phi] \equiv \widetilde{P}(g_m[\Phi],\Phi) - P_{m0} \\ &= -g_{0m}(\partial_t\Phi + qL_m[\Phi]) + O(q(q+\alpha)), \end{aligned} \qquad \text{(A10)}$$

$$L_m[\Phi] \equiv \frac{1}{2}\left[(\nabla\Phi)^2 - \left(\frac{1}{c_{m0}}\partial_t\Phi\right)^2\right]. \qquad \text{(A11)}$$

It follows from the afore-named relations that $\widetilde{P}$ is a total and real dynamical factor in the momentum equation, contrary to P^* and P_{th}. This means that the definition equation (A10) is a correct definition of potential disturbances of the total pressure in the medium. (A10) is used as the designation of the acoustic pressure P_m in a medium which is characterized by the set of equilibrium (or small signal) parameters $\{m\} \equiv \{c_{m0}, g_{0m}, a_m(\omega), ...\}$. It was shown [7] that for the solutions of the Kuznetsov equation (or (13) and (22) referred in [7]) the term $L_m[\Phi]$ is of a higher order with respect to $(\partial_t\Phi)^2$. Therefore the operator P_m in (A10) can be reduced to

$$\mathsf{P}_m = -g_{0m}\partial_t + O(q(q+\alpha)). \qquad \text{(A12)}$$

This approximation can be adequately applied for the solutions of Eq. (13) in order to determine the acoustical pressure P_m with a proper accuracy.

A2.

Applying to (22) the generalized Fourier transform [7] with respect to x and the Fourier transform with respect to the τ_2 variables, we have

$$\varsigma^2\Phi_2 + \varsigma\frac{2}{c_{20}}\omega\Phi_2 - i\frac{\omega\Phi_2}{c_{20}^2}\left[2a^2(\omega)\Phi_2 + q_2((\omega\Phi_2)\otimes(\omega\Phi_2))\right] = 0, \qquad \text{(A13)}$$

where ς is the complex wave number [7] ($\mathbf{K} = \mathbf{k}\exp(i\varphi) = \mathbf{e}_k k\exp(i\varphi) = \mathbf{e}_K K = \varsigma\mathbf{e}_k$ is the wave vector $\mathbf{e}_k = \mathbf{k}/k$, $K = (\mathbf{K}\cdot\mathbf{K}^*)^{1/2} = (\mathbf{k}\cdot\mathbf{k})^{1/2} = k$). We may rewrite (A13) in the "nearly solution form" with respect to $\varsigma\Phi_2(\omega,\varsigma)$,

$$\varsigma\Phi_2 = -\frac{\omega\Phi_2}{c_{20}}\left[1 - \sqrt{1 + i\left(\frac{2a^2(\omega)\Phi_2 + q_2(\omega\Phi_2)\otimes(\omega\Phi_2)}{\omega\Phi_2}\right)}\right], \qquad \text{(A14)}$$

$$\varsigma\Phi_2 = i(a^2(\omega)\Phi_2 + q_2(\omega\Phi_2) \otimes (\omega\Phi_2))/c_{20} + O((q+\alpha)^2), \qquad \text{(A15)}$$

that means $\varsigma^2\Phi_2 = 0 + O((q+\alpha)^2)$ and $\partial_{xx}\Phi_2(\cdot, x) = 0 + O((q+\alpha)^2)$.

A3.

To perform the factorization of the operators

$$\mathsf{w}\left[\{m\}; v^+\right] \equiv \frac{\frac{1}{2}u\cdot}{\mathsf{z}_1 + \mathsf{z}_2 + w \cdot v^+}, \qquad \text{(A16)}$$

$$\mathsf{R}_0\left[\{m\}; v^+\right] \equiv \frac{\mathsf{z}_1 - \mathsf{z}_2 + \frac{1}{2}uv^+}{\mathsf{z}_1 + \mathsf{z}_2 + wv^+}. \qquad \text{(A17)}$$

in the time domain we must solve the following inhomogeneous equation

$$\left[\mathsf{z}_1 + \mathsf{z}_2 + w \cdot v^+\right] v^- = Su\left[v^+\right]. \qquad \text{(A18)}$$

This leads to

$$-\left(z_{01}\overline{\mathcal{A}}_1 + z_{02}\overline{\mathcal{A}}_2\right) v^- + (z_{01} + z_{02} + w \cdot v^+)v^- = Su\left[v^+\right], \qquad \text{(A19)}$$

$$Su\left[v^+\right] \equiv \begin{cases} \frac{1}{2}u \cdot (\cdot) & \text{for } \mathsf{w}, \\ \left(\mathsf{z}_1 - \mathsf{z}_2 + \frac{1}{2}uv^+\cdot\right) v^+ & \text{for } \mathsf{R}_0. \end{cases} \qquad \text{(A20)}$$

If v^- may be represented (approximately) in the finite dimensionally base of the Fourier functions, then Eq. (A18) is always solvable. In this case R_0 and w are matrices, then for multi components representations they have only a numerical value. Fortunately, in the case of classical absorption (A19) is solvable in the continuos representation by means of conventional methods. In this case

$$-\left(z_{01}\overline{\mathcal{A}}_1 + z_{02}\overline{\mathcal{A}}_2\right) = \left(z_{01}\alpha_2^1 + z_{02}\alpha_2^2\right)\partial_t = \alpha z \partial_t \qquad \text{(A21)}$$

and (A19) takes the form

$$\partial_t v^- + \frac{(z_{01} + z_{02} + w \cdot v^+)}{\alpha z} v^- = \frac{Su\left[v^+\right]}{\alpha z}. \qquad \text{(A22)}$$

This is the inhomogeneous linear Bernoulli equation. Its solution (for $v^-(t=0) = 0$) has the form,

$$v^-(t) = \int_0^t \frac{1}{\alpha z} \exp\left[-\frac{1}{\alpha z}\left((z_{01} + z_{02})(t - t') + w\int_{t'}^{t} v^+(t'')dt''\right)\right] Su\left[v^+(t')\right] dt'. \qquad \text{(A23)}$$

As we see, the kernel w_0 of the fundamental operator w takes the form

$$w_0(t,t';v^+(t)) \equiv \frac{1}{\alpha z}\exp\left[-\frac{1}{\alpha z}\left((z_{01}+z_{02})(t-t') + w\int\limits_0^{t-t'} v^+(t'+\tau)d\tau\right)\right], \tag{A24}$$

and

$$\mathsf{w}\left[\,\cdot\,;v^+\right]\circ(\,\cdot\,) = \frac{u}{2}\mathsf{w}_0\circ(\,\cdot\,) = \frac{u}{2}\int\limits_0^t w_0(t,t';v^+(t'))(\,\cdot\,)dt', \tag{A25}$$

$$\mathsf{R}_0\left[\,\cdot\,;v^+\right]v^+ = \int\limits_0^t w_0(t,t';v^+(t))\left[\mathsf{z}_1-\mathsf{z}_2+\frac{1}{2}uv^+(t')\right]v^+(t')dt'. \tag{A26}$$

If $|\partial_t v^+/v^+| \ll 1/\alpha z$, then we have

$$\mathsf{R}_0 = \frac{\mathsf{z}_1-\mathsf{z}_2+\frac{1}{2}uv^+(t)}{z_{01}+z_{02}+wv^+(t)} + O(\alpha(\alpha+q))\,. \tag{A27}$$

A4.

From (112) and (114) we obtain the ratio $St_{NL,L}$ of the nonlinear v^-_{NL} and linear v^-_L velocity components (with respect to the disturbance)

$$\begin{aligned} St_{NL,L} &\equiv \frac{v^-_{NL}(t)}{v^-_L(t)} = \frac{P^-_{NL}(t)}{P^-_L(t)} + O(q(q+\alpha)) \\ &= \frac{r_v v^+(t)}{R_v} = \frac{r_p P^+(t)}{R_p} + O(q(q+\varepsilon)), \\ r_v\,,\ r_p &= O(q), \qquad P^+ = z_{01}v^+ + O(q+\alpha). \end{aligned} \tag{A28}$$

From (57) and (84) there follows the relation

$$\begin{aligned} St_{NL,L} &= \left(q\frac{2z_{01}z_{02}\left(\dfrac{z_{02}}{c_{10}}\beta_1-\dfrac{z_{01}}{c_{20}}\beta_2\right)v^+(t)}{(z_{01}-z_{02})(z_{01}+z_{02})^2}\right) \\ &= q\frac{2z_{02}\left(\dfrac{z_{02}}{c_{10}}\beta_1-\dfrac{z_{01}}{c_{20}}\beta_2\right)P^+(t)}{(z_{01}-z_{02})(z_{01}+z_{02})^2} + O(q(q+\alpha)). \end{aligned} \tag{A29}$$

To determine the above ratio by means of dimensional values, $q = 1$ is assumed. The pressure is given here in [Pa]. Let us calculate the ratio $St_{NL,L}$ for two pairs of the media:

1) (blood – heart muscle);
2) (blood – fat tissue).

The acoustic impedances $z_{...}$ and speeds of sound $c_{...}$ for the above media are equal to:

blood:

$$\begin{aligned} z_{01} &= 1060\ [\mathrm{kg/m^3}] \cdot 1567\ [\mathrm{m/s}] = 1.661 \cdot 10^6, \\ c_{10} &= 1567\ [\mathrm{m/s}], \qquad \beta_1 = 1 + 0.5 \cdot 6.05; \end{aligned}$$

heart muscle:

$$\begin{aligned} z_{02} &= 1058\ [\mathrm{kg/m^3}] \cdot 1542\ [\mathrm{m/s}] = 1.661 \cdot 10^6, \\ c_{20} &= 1542\ [\mathrm{m/s}], \qquad \beta_2 = 1 + 0.5 \cdot 5.8; \end{aligned}$$

fat tissue:

$$\begin{aligned} z_{02} &= 920\ [\mathrm{kg/m^3}] \cdot 1476\ [\mathrm{m/s}] = 1.352 \cdot 10^6, \\ c_{10} &= 1476\ [\mathrm{m/s}], \qquad \beta_2 = 1 + 0.5 \cdot 11. \end{aligned}$$

In the first case (blood – heart muscle) we have $|St_{NL,L}| = 0.11 \cdot 10^{-9}[1/\mathrm{Pa}] \cdot P^+$, hence for P^+=10^7 [Pa] one obtains the value $St_{NL,L} = 0.0011$. In the second case (blood – fat tissue), $|St_{NL,L}| = (3.7 - 4.1) \cdot 10^{-9}[1/\mathrm{Pa}] \cdot P^+$, hence for $P^+ = 10^7$ [Pa] one obtains the value $St_{NL,L} \approx 0.037$. In this way it was found that the reflection caused by the nonlinearity parameter only is a small value. However it is more then 30 times higher in the second case (neglecting the reflection caused by differences in the acoustic impedances).

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