Nonsingular Meshless Method in an Acoustic Indoor Problem

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An efficiency of the nonsingular meshless method is analyzed in an acoustic indoor problem. The solution is assumed in the form of the series of radial bases functions. The Hardy's multiquadratic functions, as the bases, are taken into account. The room acoustic field with uniform, impedance walls is considered. The representative, rectangular cross section of the room is chosen. Practical combinations of acoustic boundary conditions, expressed through absorption coefficient values, are considered. The classical formulation of the boundary problem is used. It is established any coefficient in the multiquadratic functions depend on the number of influence points, the frequency and the absorption coefficient. All approximate results are calculated in relation to the exact ones. This way, it is proved that the meshless method based on the multiquadratic functions is simple and efficient method in the description of the complicated acoustic boundary problems for the low and medium ranges of frequency.

Keywords: architectural acoustics; meshless method; radial bases functions; impedance boundary condition.

1. Introduction

One of the main aims of room acoustics is to describe acoustical field. Both geometry of the room and the sound absorption of walls determine the acoustic phenomena (KUTTRUFF, 2000; PILCH, KAMISIŃSKI, 2011; RUBACHA *et al.*, 2012; KAMISIŃSKI, 2012; KAMISIŃSKI *et al.*, 2016).

Three approaches to room acoustics modeling are distinguished: image source methods (ISM) (based on the geometrical theory of the diffraction), acoustics energy methods (AEM) (MEISSNER, 2013) and wave-based methods (WBM) (MEISSNER, 2009; 2016b; KAMISIŃSKI *et al.*, 2016; SILTANEN, 2010). Each approach has advantages and disadvantages and different approaches have some links between them (RINDEL, 2010).

The ISM are widely applicable, but these methods lack diffraction and they may be applied in medium and high frequencies rather. Furthermore, they require too much computational resources for higher-order reflections. The AEM lack phase information, so they are not efficient at the early part of the response and at the low frequency. The WBM are computationally demanding and for this reason they are efficient at low frequencies.

In practice hybrid methods are also implemented, where two or three different methods, one belonging to each groups mentioned above, can be combined. The links among methods could be utilized to combine the results for full room response, that would lead to both efficient and accurate room acoustics modeling.

Generally speaking, the WBM ought to be used for low frequencies, the ISM for the early part of the room response for middle and high frequencies and the AEM for the rest of the response for middle and high frequencies. Searching of the general and all-purpose method is an open question. Such a method ought to be searched among WBM, i.e. particularly among finite element methods, boundary element methods and meshless methods.

The finite elements methods (FEM) is widely applied in the interior domain technical problems (RAO, 2005; FISH, BELYTSCHKO, 2007; DOBRUCKI *et al.*, 2010). In the FEM, both the domain and the boundary of the problem must be discretized.

The boundary element method (BEM) has been recognized as an efficient numerical tool for exterior domain problems (BEER *et al.*, 2008; MANOLIS, POLYZOS, 2009). This is because the physical boundary needs to be discretized only. In a standard, the base functions in BEM constitute fundamental solu-

tions (FS), which satisfy the governing equation. However, this BEM needs costly numerical calculations of singular and nearly singular integrals (SLADEK, SLADEK, 1998; SLADEK *et al.*, 2000). Recently, the BEM, based on the Trefftz functions (they are nonsingular), is developed in (BRAŃSKI *et al.*, 2012; BORKOWSKI, 2015; BRAŃSKI, BORKOWSKA, 2015a; 2015b). Unfortunately, the construction of the mesh of the boundary in the BEM is a bit difficult.

For this reason, recently researches are concerned on meshless methods (MLM), sometimes called meshfree method (MFM) (ATLURI, 2004; CHEN *et al.*, 2013; BRAŃSKI, PRĘDKA, 2014; PRĘDKA, 2015).

Generally, boundary MLM is classified into weak and strong categories (group of methods); an excellent survey is given in (FU *et al.*, 2014). Both groups include many methods. The strong group includes, among other the method of fundamental solutions (MFS) (CHEN *et al.*, 2000; YOUNG *et al.*, 2006); so far, the MFS plays a major part.

The basic concept of the MLM is to express of the solution of the partial differential equation (PDE) by the series, in which the base constitutes the radial bases functions (RBF) and coefficients, which are interpreted as intensities of influence points. Unknown coefficients can be obtained by collocation of the boundary conditions or minimizing of the boundary error functional (averaged error) (PAWŁOWSKI, 2009). The number and the distribution of influence points are significant in the MLM and they remain the open issues. Up to now, the great effort has been made to solve both problems.

The goal of this study is to propose the non-singular MLM, one of the WBM, to the solution of the acoustic indoor problem with impedance boundary conditions imposed on the walls. This method is tested in the wide range of audible frequencies. Multiquadratic functions are chosen as the RBF. It comes down to expression the parameter of RBF as a function of number of influence points, the frequency and the absorption coefficient. All considerations are supported by many numerical experiments. The approximated results are compared to the exact ones.

2. Two dimensional (2D) boundary acoustic problem

Let be given the 2D acoustic boundary problem in the rectangular domain Ω. The mathematical model is described by 2D wave equation and Robin boundary conditions,

$$D^2 u(\mathbf{x}, t) - (1/c^2)D_t^2 u(\mathbf{x}, t) = f_e(\mathbf{x}, t),$$
$$\mathbf{x} = \mathbf{x}' \in \Omega, \quad (1)$$

$$\eta(\mathbf{x})D_{\mathbf{n}} u(\mathbf{x}, t) + \gamma(\mathbf{x})u(\mathbf{x}, t) = g(\mathbf{x}, t),$$
$$\mathbf{x} \in \Gamma, \quad (2)$$

where the initial conditions are assumed equal to zeros, c is the speed of sound in that medium; $\mathbf{x} = (x, y)$; t – time; Ω – physical domain; Γ – boundary of the domain; $u(\mathbf{x}, t)$ – acoustic potential, $f_e(\mathbf{x}, t)$ – excitation of acoustic field; more precisely, in 2D, it is the cross section of the harmonic pulsating line acoustic source, which may be constitutes by the cylinder with a very little radius; $g(\mathbf{x}, t)$ – given function, $\gamma(\mathbf{x})$, $\eta(\mathbf{x})$ – given functions, $\mathbf{n}$ – unit normal vector pointing outward.

Let

$$f_e(\mathbf{x}, t) = F(\mathbf{x}) \exp(i \omega_f t), \quad (3)$$

$$u(\mathbf{x}, t) = U(\mathbf{x}) \exp(i \omega_f t), \quad (4)$$

where $i = \sqrt{-1}$.

Substituting these expressions to Eqs. (1) and (2) leads to (WU *et al.*, 2011)

$$\mathsf{L}U(\mathbf{x}) = \Delta U(\mathbf{x}) + k_f^2 U(\mathbf{x}) = F(\mathbf{x}),$$
$$\mathbf{x} = \mathbf{x}' \in \Omega, \quad (5)$$

$$\mathsf{B}U(\mathbf{x}) = \eta(\mathbf{x})D_{\mathbf{n}}U(\mathbf{x}) + \gamma(\mathbf{x})U(\mathbf{x}) = G(\mathbf{x}),$$
$$\mathbf{x} \in \Gamma, \quad (6)$$

where k_f is the wave number, $k_f = \omega_f/c$, ω_f – angular frequency, $F(\mathbf{x})$, $G(\mathbf{x})$ – given functions; the rest of symbols is given in Fig. 1.
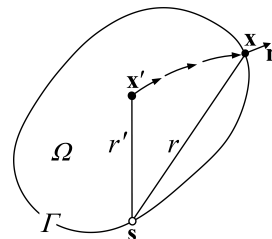

Fig. 1. Geometry of the boundary problem.

From the Robin boundary condition, the Neumann boundary condition can be derived, i.e. $\eta(\mathbf{x}) = 1$, $\gamma(\mathbf{x}) = 0$ and $G(\mathbf{x}) = 0$, or the Dirichlet boundary condition, i.e. $\eta(\mathbf{x}) = 0$, $\gamma(\mathbf{x}) = 1$ and $G(\mathbf{x}) = 0$.

2.1. Acoustic boundary conditions

In acoustics, the Robin boundary condition corresponds to the specifying surface acoustic impedance (PRĘDKA, 2015; KOCAN, BRAŃSKI, 2015)

$$z(\mathbf{x}) = p(\mathbf{x})/v(\mathbf{x}), \quad (7)$$

where $z(\mathbf{x})$ is an acoustic impedance, $p(\mathbf{x})$ – acoustics pressure; $v(\mathbf{x})$ – particle velocity. Sound parameters described by the acoustic potential $U(\mathbf{x})$ take the forms

$$p(\mathbf{x}) = i \rho \omega U(\mathbf{x}), \quad (8)$$

$$v(\mathbf{x}) = -D_{\mathbf{n}}U(\mathbf{x}) = -\operatorname{grad} U(\mathbf{x}), \quad (9)$$

where ρ is the air density.

As it can be seen, the impedance boundary condition in Eq. (7) can be written in the form

$$z(\mathbf{x})D_{\mathbf{n}}U(\mathbf{x}) + i\rho\,\omega\,U(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Gamma. \qquad (10)$$

Hereunder this boundary condition is considered.

2.2. Acoustic boundary conditions as a function of absorption coefficient

In practice, the acoustic impedance $z(\mathbf{x})$ is in fact the acoustic impedance of any material and it is won via the measure of the absorption coefficient $\alpha(\mathbf{x})$ (Meissner, 2016a; Piechowicz, Czajka, 2012). There are several methods to measure the $\alpha(\mathbf{x})$ or the acoustic impedance of acoustic materials. The classification of them is given in (Gerai, 1993). Both $\alpha(\mathbf{x})$ and $z(\mathbf{x})$ are connected each other by the formula (Kuttruff, 2000)

$$z(\mathbf{x}) = \rho\,c\,\frac{1 + (1 - \alpha(\mathbf{x}))^{1/2}}{1 - (1 - \alpha(\mathbf{x}))^{1/2}}. \qquad (11)$$

The real part of $z(\mathbf{x})$ is considered only, hereby an angle between an acoustic pressure and the particle velocity is assumed as zero. Thereby, all numerical results are qualitative.

On account of Eq. (6), instead of Eq. (10), is

$$D_{\mathbf{n}}U(\mathbf{x}) + z_0(\mathbf{x})U(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Gamma, \qquad (12)$$

where $z_0 = (\omega\,\rho)/z(\mathbf{x})$.

For the practical acoustic case the hard floor is modeled through the Neumann boundary condition (N), but the walls and ceiling are modeled through impedance Robin boundary conditions (R), Fig. 2 where $\mathbf{x}_0$ is the place of the source, $\mathbf{x}_i \in \Omega$ – arbitrary point, $r_i = |\mathbf{x}_0 - \mathbf{x}_i|$. So, in the following, one has,

$$D_{\mathbf{n}}U(\mathbf{x}) = 0, \qquad \mathbf{x} \in N, \qquad (13)$$

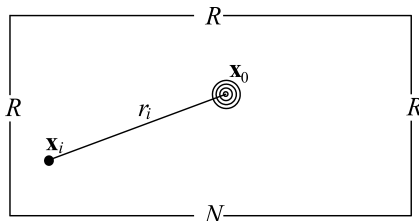$$D_{\mathbf{n}}U(\mathbf{x}) + z_0(\mathbf{x})U(\mathbf{x}) = 0, \qquad \mathbf{x} \in R. \qquad (14)$$

Fig. 2. Geometry of the acoustic problem.

3. Radial basis functions (RBF)

From historical point of view, the RBF are introduced to the numerical methods in (Kuttruff, 2000),

though this name is not used there; this name is introduced in (Kansa, 1990). Theoretical bases of the RBF are worked out in (Buhman, 2004; Chen *et al.*, 2007; Fasshauer, 2010; Ling, 2003). Furthermore, except for the definition, the convergence of the solution formed on their base and an error of this solution are estimated. The definition of the RBF is formulated in (Chen *et al.*, 2007; Fasshauer, 2010; Ling, 2003). The $\mathsf{R}(r_\nu)$ depend only on $r_\nu = (\mathbf{s}_\nu^2 - \mathbf{x}^2)^{1/2}$, where r_ν is a distance between an influence point of the physical effect $\mathbf{s}_\nu \equiv \mathbf{x}_{s\nu}$ (it may be recognized as a source) and the current point $\mathbf{x}$ (Kansa *et al.*, 2009).

The RBF make up two main groups:

- the former, the RBF do not satisfy neither the differential equation nor boundary conditions; in order to not proliferate notations, hereunder these RBF are marked by $\mathsf{R}(r)$ and they are basis functions in the domain-boundary group of the MLM,

- the latter, the RBF satisfy the differential equation; they are either the Trefftz functions $u^\circ(r)$ or the fundamental solution $u^\bullet(r)$; these RBF are basis functions in the boundary group of the MLM.

The main advantages of the RBF are:

- they provide good results at the low cost of calculations,

- if they are globally defined, they do not generate of the rarely main matrix (Franke, Schaback, 1998),

- they assure the exponential convergence of the solution.

However, the RBF provide the smooth solutions only, so that to achieve the singular solution, any functions ought to be added to the RBF; as a rule, there are either fundamental solutions (Li *et al.*, 2008), or the polynomial of the appropriate order (Cheng, 2000; Powell, 1992).

In the following, the RBF of the first group are enumerated, since they play in technique a major part: Hardy's multiquadratic (Hardy, 1971), inverse multiquadratic, Gauss', Duchon's (Duchon, 1976; Cheng, 2000), Wendland's and others.

It is proved in the paper (Franke, 1982), that Hardy's multiquadratic $\mathsf{R}(r)$ are the best accurate to the multidimensional interpolation of discrete values. Furthermore, the convergence of interpolating series is theoretically proved in (Buhman, 2004). So, in the following, the multiquadratic $\mathsf{R}(r)$ are analyzed in the indoor acoustic boundary problem. They are in the form

$$\mathsf{R}(r) = (-1)^{\lceil\beta\rceil}(C^2 + r^2)^\beta,$$
$$C > 0, \qquad \beta > 0, \qquad \beta \notin N, \qquad (15)$$

where $\lceil\beta\rceil$ means the smallest integer larger than β.

To make the $\mathsf{R}(r)$ useful, the coefficient C must be determined.

4. Domain-boundary MLM

In this group of methods, the solution is assumed as the series, here

$$\widetilde{U}(\mathbf{x}') = \sum_{\nu} a_\nu \mathsf{R}(r'_\nu), \qquad r'_\nu = |\mathbf{s}_\nu - \mathbf{x}'|, \qquad (16)$$

where $\mathbf{s}_\nu \in \overline{\Omega} = \Omega \cup \Gamma$, $\mathbf{x}' \in \Omega$, Fig. 3a.
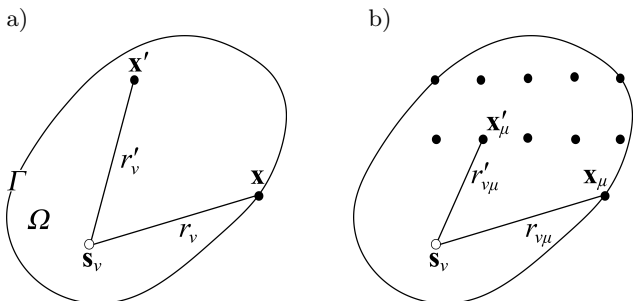

Fig. 3. a) Geometry of the problem, b) geometry of the discrete problem.

The base $\mathsf{R}(r_\nu)$ does not satisfy the differential equation, so it is made up the domain-boundary group of methods. In Eq. (16), the coefficients a_ν are staying to calculation. For this purpose, first the solution (16) is substituted to the problem (5) and (6), hence

$$\sum_{\nu} a_\nu \mathsf{LR}(r'_\nu) = F(\mathbf{x}'), \qquad r'_\nu = |\mathbf{s}_\nu - \mathbf{x}'|, \quad (17)$$

$$\sum_{\nu} a_\nu \mathsf{BR}(r_\nu) = G(\mathbf{x}), \qquad r_\nu = |\mathbf{s}_\nu - \mathbf{x}|. \quad (18)$$

Next, to set up the discrete problem, the set of collocation points $\{\mathbf{x}_\mu\}$ should be selected in the domain $\overline{\Omega}$, where $\mu = 1, 2, ..., m = n$. In detail, the collocation points in the domain Ω may be formed by $\mathbf{x}'_\mu$, i.e. $\mathbf{x}'_\mu \in \Omega$, and the collocation points on the boundary Γ may be marked by $\mathbf{x}_\mu$, $\mathbf{x}_\mu \in \Gamma$, Fig. 3b. Considering the set $\{\mathbf{x}_\mu\}$ in Eq. (17) and (18), one obtains

$$\sum_{\nu} a_\nu \mathsf{LR}(r'_{\nu\mu}) = F(\mathbf{x}'_\mu), \qquad r'_{\nu\mu} = \left|\mathbf{s}_\nu - \mathbf{x}'_\mu\right|, \quad (19)$$

$$\sum_{\nu} a_\nu \mathsf{BR}(r_{\nu\mu}) = G(\mathbf{x}_\mu), \qquad r_{\nu\mu} = |\mathbf{s}_\nu - \mathbf{x}_\mu|. \quad (20)$$

5. Discrete acoustic domain-boundary MLM

For 2D acoustic problem, the differential operator is $\mathsf{L} = \Delta + k^2 = D_x^2 + D_y^2 + k^2$. Hence, instead of the Eq. (19) one has

$$\sum_{\nu} a_\nu \left(D_x^2 \mathsf{R}(r'_{\nu\mu}) + D_y^2 \mathsf{R}(r'_{\nu\mu}) + k^2 \mathsf{R}(r'_{\nu\mu})\right) = F(\mathbf{x}'_\mu). \quad (21)$$

The derivatives $D_x^2(\cdot)$ and $D_y^2(\cdot)$ need the explanation; since $r'_{\nu\mu} = \left|\mathbf{s}_\nu - \mathbf{x}'_\mu\right|$, then in $D_x^2(\cdot)$, the derivative with respect to x should be understand as derivative with respect to x'_μ and so on.

Whereas the Neumann and Robin boundary conditions are given respectively by

$$\sum_{\nu} a_\nu D_\mathbf{n} \mathsf{R}(r_{\nu\mu}) = 0, \quad \mathbf{x}_\mu \in N, \quad (22)$$

$$\sum_{\nu} a_\nu (D_\mathbf{n} \mathsf{R}(r_{\nu\mu}) + z_0(\mathbf{x}_\mu)\mathsf{R}(r_{\nu\mu})) = 0, \quad \mathbf{x}_\mu \in R. \quad (23)$$

In Eqs. (22) and (23), the versor $\mathbf{n}$ is defined at $\mathbf{x}_\mu$, it is perpendicular to the boundary Γ and directed outside the domain Ω, for example, if $\mathbf{x}_\mu \in N$, the $D_\mathbf{n}(\cdot) = -D_y(\cdot)$ and so on.

6. Numerical calculations, results, conclusions

In practice, instead of the acoustic potential, the acoustic pressure plays a major part. First of all, the acoustic pressure via Eq. (8) is calculated and consistently, the value of the sound pressure level at the point $\mathbf{x}$ is given by

$$L(\mathbf{x}) = 20 \log |p(\mathbf{x})/p_0|, \qquad (24)$$

where $p_0 = 2 \cdot 10^{-5}$ Pa.

To notice quantitative change of $L(\mathbf{x})$, first the mean value of the acoustic pressure p_m ought to be calculated based on the equation,

$$p_m = 1/n_i \sum_i p(\mathbf{x}_i), \qquad (25)$$

where $i = 1, 2, \ldots, n_i$ number of calculated points inside of the acoustic room.

Next, instead of $L(\mathbf{x})$, one has the mean value of sound pressure level in the room, i.e. $L_m = 20 \log |p_m/p_0|$; hereunder this quantity is considered.

An acoustic source in 2D is represented by the source function $F(\mathbf{x})$ in Eq. (5). In an explicit form and in steady state the $F(\mathbf{x})$ constitutes the solution of the radial part of the Bessel's differential equation (MCLACHLAN, 1964). Here, the 0-order, Hankel function of the second kind plays the major part, hence $F(\mathbf{x}) = A H_0^{(2)}(k_f r)$, where A is an intensity of the source. This is because it describes the outward propagating wave solution of the Bessel's equation.

The intensity A is chosen, so that the L_m takes the same value for different values of the absorption coefficient α and frequencies f. This way it is easy to track the changeability of the L_m as the function of number of influence points n, α and f.

Hereafter, numerical details are presented for discrete values of the full scope of the absorption coefficient, i.e. $\{\alpha\} = \{0.1,$ step $0.1, 0.9\}$ and chosen frequencies, namely $\{f\} = \{250, 500, 1000, 2000\}$ Hz. Hence, equivalent lengths of waves are $\lambda = \{1.376, 0.688, 0.344, 0.172\}$ m respectively. The set number of influence points on the length of the wave $\{n/\lambda\}$ is $\{n\} = \{4, 5, 6, 7\}$.

The following global values and symbols are assumed: $\rho = 1.205 \, \text{kg/m}^3$, $c = 344 \, \text{m/s}$, $\{a_x, b_x\} = \{0, 5\} \, \text{m}$, $\{a_y, b_y\} = \{0, 2.5\} \, \text{m}$, the point forced source is placed at the point $\mathbf{x}_0 = \{x_0, y_0\} = \{2.5, 1.25\} \, \text{m}$. Furthermore, the assumptions $z_0(a_x) \equiv z_0(b_x) \equiv z_0(b_y) = Z$ are the most frequently appears in acoustic, where e.g. $z_0(a_x)$ is $z_0(x)$ on the edge a_x. Otherwise, $z_0(a_y) = \infty$ as a result of the Neumann boundary condition. Additionally, the influence points are marked by „o" and the collocation points, marked by „•". Both kinds of points cover each other and they and the rest of labels are depicted in Fig. 4.
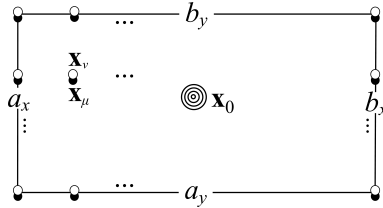

Fig. 4. Distribution of all points in the $\overline{\Omega}$.

Hereunder, all results are related to the exact ones, which are presented in (BRAŃSKI *et al.*, 2017), hence the rest assumptions are made for this paper. It is done in the following manner. For current n, α and f in MLM, the coefficient C in the multiquadratic functions is searched in order to the L_m calculated by approximate way, should be the same as the exact $L_{e;m}$, namely $L_m = L_{e;m}$; the results are shown in Fig. 5.

To attain the aim of the paper, three kinds of calculations are carried out.

1) First, using results to Fig. 5, the coefficient C is expressed as a function of the absorption coefficient $\alpha(\mathbf{x})$ and different values of the frequency f: $C = C(\alpha, f)$, $n = 5$; results are depicted in Fig. 6 As can be seen, the coefficient C does not depend on the absorption coefficient $\alpha(\mathbf{x})$, but it strongly depends on the frequency f. But this relationship is predictable, so it may be described by formula. Another presentation of similar results, for all values n, is given below, in Fig. 7.
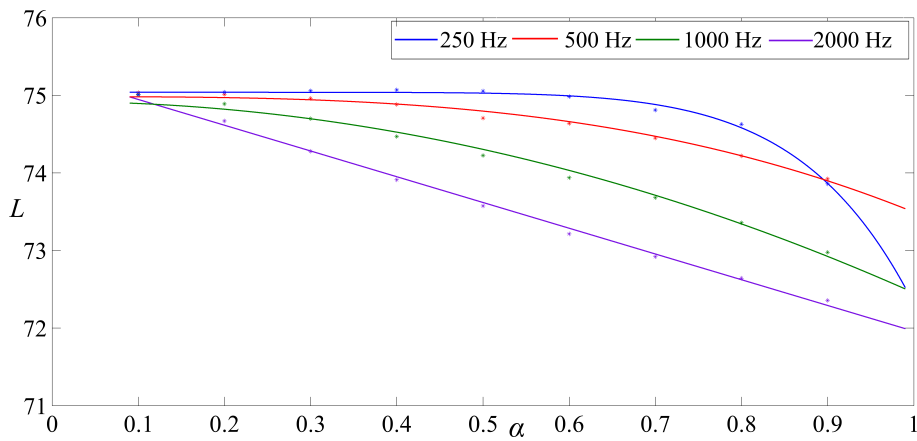

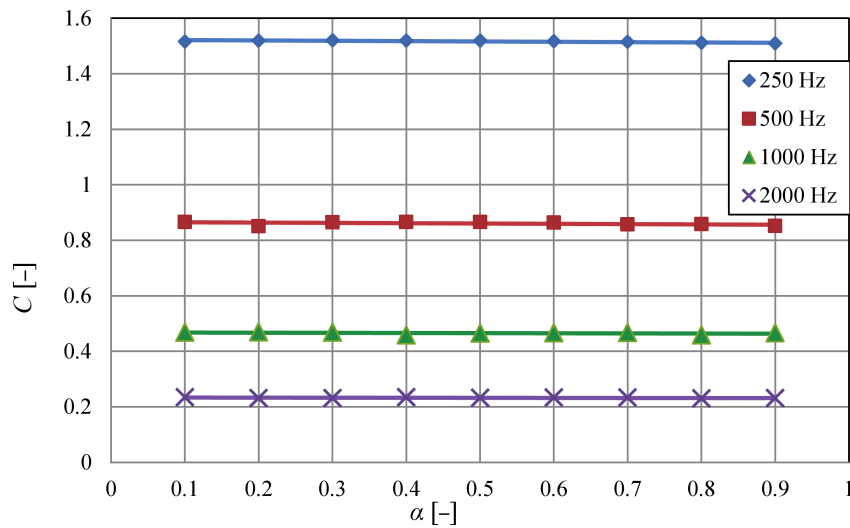Fig. 5. Sound pressure level $L_{e;m} = L_m = L_m(\alpha, f)$, $n = 5$.
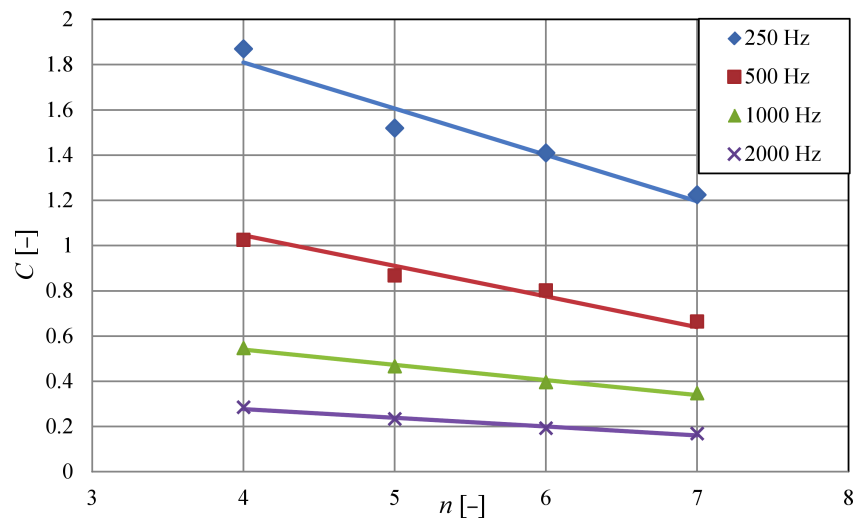

Fig. 6. Parameter $C = C(\alpha, f)$, $n = 5$.

Fig. 7. Parameter $C = C(n, f)$, $\alpha = 0.5$.

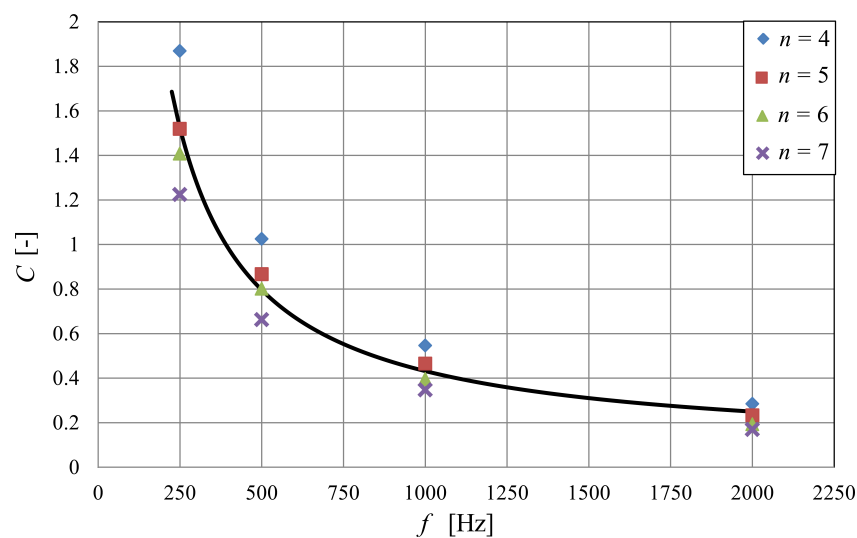
Fig. 8. Parameter $C = C(f)$, $\alpha = 0.5$.

2) In this stage, the coefficient C is calculated as a function of number of influence points n and different values of the frequency f: $C = C(n, f)$. The value of the absorption coefficient α is assumed as constant, i.e. $\alpha = 0.5$; the results are presented in Fig. 7. It should be noted that the coefficient C almost linearly depends on the number of influence points n, but it strongly depends on the frequency f again. Straight lines $C = a_1\, n + a_2$ are given in Fig. 7, where $\{a_1, a_2\}$ are any constants and they are calculated based on the known discrete values of C and least squares approximation theory.

3) At the end, the coefficient C is calculated as the function of the frequency, $C = C(f)$ for particular number $n = \{4, 5, 6, 7\}$, $\alpha = 0.5$; the results are presented in Fig. 8. At first side the curves are in the shape of hyperbolic. Since, for all n, they are very close to each other, so they may be replaced by one hyperbolic function. This function

takes the form $C = b_1/f + b_2$, where $\{b_1, b_2\}$ are any constants and they are calculated based on the known discrete values of C and least squares approximation theory; the result is given in Fig. 8 and $b_1 = 364.179$, $b_2 = 0.0673$.

7. Conclusions

The paper proposes the meshless method (MLM), based on Hardy's multi-quadratic functions, to the solution of the acoustic boundary problem with uniform impedance boundary conditions imposed on the walls. The coefficient C, occurring in the radial basis function (RBF), can be expressed analytically as a function of the separate quantities. Hence, some conclusions are arisen:

1. The coefficient C does not depend on the absorption coefficient α.

2. The coefficient C almost linearly depends on the number of influence points n: for lower frequency this dependence is higher, but for higher frequency the C is almost not depend on n.

3. The coefficient C strongly depend on the frequency f. Fortunately, this relationship can be expressed analytically, i.e. for all n, it may be described in the shape of any hyperbolic function.

4. This MLM is restricted to low and middle frequencies.

To sum up, the MLM, based on Hardy's multiquadratic functions, is very handy and efficient to the solution of the indoor acoustic boundary problem with uniform impedance boundary conditions imposed on the walls. For future, an efficiency of the MLM in over mentioned form, ought to be researched to the solution of the 3D acoustic boundary problem and to the solution 2D and 3D acoustic boundary problems with non-uniform (random) impedance boundary conditions. This method ought to be especially useful to such problems, which may not be solved via exact methods, for example with complicated geometry.

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