ARCHIVES OF ACOUSTICS
20, 4, 311–326 (1995)

ACOUSTIC MODELLING OF SURFACE SOURCES. PART I. PISTON MODEL, DISCRETIZING ERROR, AXISYMMETRICAL PROBLEM

A. BRAŃSKI

Institute of Technology
Pedagogical College
(33-310 Rzeszów, ul. Rejtana 16a)

The paper discusses the application of irregular boundary elements in BEM method. These elements were proposed in order to reduce the number of the elements in acoustical model of surface source. The boundaries of irregular elements were introduced by optimizing a mean square distance between exact directivity function and approximation one. The constant vibration velocity on each element was calculated employing the definition of an average function value. For simplicity as an example radiation axisymmetric problem governed by Helmholtz–Rayleigh integral was considered. Examples are shown that applying irregular discretization better model may be obtained.

The list of symbols

a, b — internal and external radius of surface S_1,
$J_0(x)$ — Bessel function of the first kind and zeroth order,
k — dimensioness wave number,
G — fundamental solution to the Helmholtz equation in the space E_3,
S_1 — vibrating surface (driving surface),
S_0 — baffle board (baffle),
v — vibration velocity of the driving surface,
Φ — velocity potential,
ρ, ϕ — polar coordinates.

1. Introduction

Boundary Element Method (BEM) is now a method frequently applied to solve the boundary value problem in regions with irregular surfaces [10]. This method is also widely applied to the problems of sound radiation by a sources with a finite baffle (source = vibrating surface + baffle), or sound diffraction on irregular surfaces [8, 11].

The mathematical model describing the physical model both of those problems includes, among others, boundary integral equation (BIE). There is a short review of BIE applied to describe harmonic acoustic phenomena in time given in [6].

In the problem of sound radiation by a source with a finite baffle the solution is given by the Helmholtz–Huygens integral, where the acoustic pressure acting on the surface of the source, as well as the vibration velocity of the vibrating surface, are unknown. Both acoustic parameters are coupled with each other by a BIE. The vibration velocity in selected points of the source can be determined experimentally, whereas the acoustic pressure is the solution of BIE. Practically, the application of numerical methods is the only possible way to solve these equations. Then, after the discretization, the integral equation is replaced by a system of algebraic equations.

The number of unknowns and constans in the system of equations is proportional to the number of discrete elements. Too many discrete elements complicate the physical model causing it's mathematical equations to be difficult in solving. Therefore it is evident that the elaboration of a model having the minimum number of elements and giving a good result (an optimal model) should be aimed at. The purpose of this paper is to work out a model with minimal number of the discrete elements which ensures the theoretical predictions to fit in with experimental observations and measurements.

The discretization including source geometry and acoustic parameters should be the first step in modelling.

In the examples of radiation and diffraction problems analysed and published so far a constant value of acoustic parameters on a source of an arbitrary geometry was assumed. Then, above all, the fundamental discretization is imposed by the source geometry (e.g., the boundaries). An additional regular discretization, if needed, is carried out between the lines of the fundamental discretization. The division to smaller elements is carried out until a good model is approached. The model obtained using this method, however, includes too many elements.

In this paper, addition irregular discretization was proposed in order to reduce the number of model elements. In the literature concerning sound radiation and diffraction, any additional irregular discretization of the source (any optimal model problem) has not been discussed yet.

According to the publication [5], it appears that the regular discretization and all irregular ones do not have any theoretical base.

Systematization of the difficulties appearing in source modelling problems (including discretization) and providing the rules of finding their solutions is taken up in the publication series Acoustic Modelling of Surface Source. This paper is the first one in this series.

The irregular discretization problem is examined basing on the example of a plane source where the vibration velocity function is not constant. The problem may be simplified due to the fact that the discretization of the source geometry is no longer needed and only the vibration velocity has to be discretized. The mathematical description is also simplified. This case is not described by Helmholz–Huygens integral with BIE but only by Helmoholz–Rayleigh integral.

In general, plane surface source modelling is carried out in the following stages:

1 — the vibrating surface is discretized into $j=1, 2,\dots,J$ elements;

2 — vibration velocity points (i-points) are selected on each element, $i=1, 2,\dots,I$;

3 — basing on the measured discrete values of the vibration velocity the vibration velocity on each element is evaluated in one of the three ways:

3.1 — as the arithmetic mean of vibration velocity values measured in a large number of the i-points. Then it is assumed that every point of the given element is vibrating with the arithmetic mean vibration velocity. It is assumed, therefore, that the element vibrates in the same way as a piston. A set of all pistons generates a piston model of the source (PM^m). It has to be observed that the measurement of the vibration velocity values in larger number of i-points will not affect significantly to the value of the average vibration velocity. The quality of the model created in this way results from only the discretization error, i.e. the error resulted from the insufficient large number of i-points is not be included in the model.

3.2. — as the arithmetic mean of the values of the vibration velocity measured in the finite number of i-points.
Similarly as in item 3.1., a piston model (PM^α) is considered. The quality of this model is caused by the discretization error as well as by the insufficiently large number of the i-points.

3.3. — an appropriate shape of vibration velocity function is reproduced making use of interpolation functions. The source elements with such built vibration velocity functions constitute so called the interpolation model (IM).

For the assumed vibration velocity function, the convergence of the model with the source can be analysed. It can be done by examining the acoustic fields calculated from the Helmholz–Rayleigh integral and from "discrete" form of this integral.

In acoustic the divergence of the model with the exact one can be defined, for example, by the distance between their directivity functions (DFs) i.e.:

1 — an absolute distance;
2 — a relative distance;
3 — a mean square distance.

Applying the second or the third option and the irregular discretization a minimal relative or mean square distance is sought for the given number of elements. The problems outlined above belong to the optimization problems [12]. Therefore, the source model obtained as a solution of these problems is called the optimal model.

In this paper, a plane axisymmetric source was considered where the vibration velocity function is axially symmetrical, the source axis (the z-axis on Fig. 1) being the axis of symmetry of the velocity function. A line of the source is the results of a cross section of such source made by a perpendicular plane. The discretization of the source reduces to the discretization of the line of the vibration velocity function. A vibration velocity function without nodes was assumed on the vibrating surface. The value of the constant vibration velocity was calculated as the average value of the vibration velocity function (see item 3.1). Applying the irregular discretization, the minimal number of elements was looked for by the analysis of the absolute distance between the exact DF and DF of the model.

2. Acoustic field of an axisymmetric source

The acoustic field of a circular vibrating surface located in an infinite and rigid baffle is calculated using Hyugens – Raleigh integral [9]

$$\Phi_{MN}=\frac{1}{2\pi}\int_S v_{MN}(\rho,\phi)G(r)dS, \tag{2.1}$$

where in polar coordinates

$$dS=\rho\, d\rho\, d\phi, \tag{2.2}$$

$$G(r)=\frac{\exp[-\mathrm{ikr}(\rho,\phi)]}{r(\rho,\phi)}, \tag{2.3}$$

Vibration velocity of an axisymmetric source will be denoted by $v_{0N}(\rho)$. Such source radiates an axisymmetric acoustic field, constant for each constant angle ϕ_P and ρ_P. This field can be considered in a plane crossing the source axis of symmetry and described by angle ϕ_P=const. Acoustic field of the axisymmetric problem is described by formula (1) for $M=0$ and for e.g. $\phi_P=0$, that is

$$\Phi_{ON}=\frac{1}{2\pi}\int_\rho v_{ON}(\rho)\left[\int_0^{2\pi} G(r)d\phi\right]\rho\, d\rho. \tag{2.4}$$

It is possible to calculate analytically the integral inside the square bracket. Thus the evaluation of the axisymmetric field reduces to calculation of the integral (2.4) only

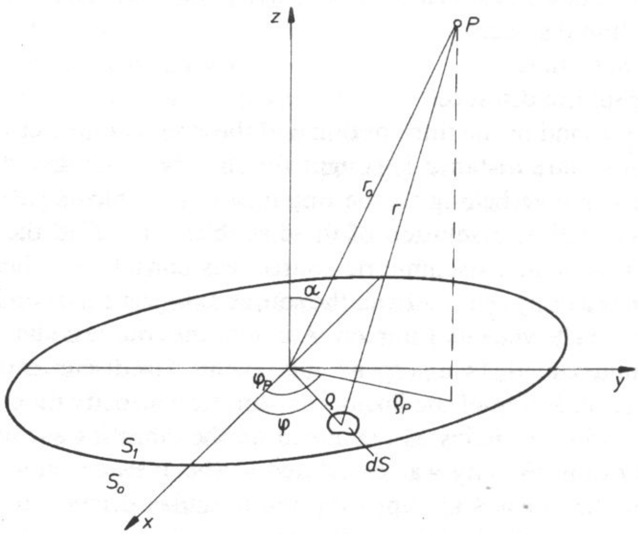


Fig. 1. The geometry of the problem.

over the source cross section. However the calculation of the integral requires the separation of integration variables ρ, ϕ by expanding the integrand in a series of special functions [7]. To calculate the integral in bracket it is convenient to apply the numeric integration.

Formula (2.4) is valid for any distance of point P from the source. But only the far field was considered in the present publication. In the far field the Fraunhofer approximation is applied to formula (2.4), Fig. 1. In the far field:

$$\frac{1}{r} \cong \frac{1}{r_0}, \quad r \cong r_0 - \rho\cos(\rho, r_0) = r_0 - \sin\alpha\cos(\phi - \phi_P). \tag{2.5}$$

Inserting (2.5) into (2.3) and then to (2.4) one gets

$$\Phi_{0N}(a) = C_0 Q_{0N}(\alpha), \tag{2.6}$$

where

$$C_0 = \frac{\exp(-ikr_0)}{r_0}, \tag{2.7}$$

$$Q_{0N}(\alpha) = 2\pi \int_L v_{0N}(\rho)\left[\int_0^{2\pi} \exp(ik\rho\sin\alpha\cos\phi)\,d\phi\right]\rho\,d\rho. \tag{2.8}$$

The integral in the square bracket is given by the following formula [7]

$$\int_0^{2\pi} \exp(ik\rho\sin\alpha\cos\phi)\,d\phi = 2\pi J_0(k\rho\sin\alpha). \tag{2.9}$$

Therefore,

$$Q_{0N}(\alpha) = \int_\rho v_{0N}(\rho)\, J_0(k\rho\sin\alpha)\rho\,d\rho \tag{2.10}$$

Formula (2.6) expresses the acoustic potential in the far field and is the starting point for the calculation of the cross section of the DF.

The DF is defined as a ratio of a field radiated in any direction to the field (2.7) radiated by a point source located in the center of the coordinate system (2.7). Therefore, the DF is given by formula (2.10)

3. Acoustic field of the piston model

3.1. The average value of the vibration velocity function

A constant vibration velocity on j-element was calculated employing the definition of average function value which, in polar coordinates, is given by the following formula:

$$v_j^m = \frac{1}{L_j} \int_{G_1}^{G_2} v_{0N}(\rho)\,d\rho, \tag{3.1}$$

where G_1, G_2 are boundaries of the discrete j-element and

$$L_j = G_2 - G_1. \tag{3.2}$$

This formula should be considered as the arithmetic mean of vibration velocity calculated from the velocities measured in large number of i-points on the vibrating surface (see "Introduction", item 3.1).
If the discretization is irregular the boundaries G_1, G_2 of the j-element are chosen arbitrary. However, if any of the j-elements (or the whole vibrating surface) having the boundaries G_1, G_2, is divided into $j = 1, 2, ..., J$ smaller elements equal to each other than the new boundaries are given by formula

$$\begin{aligned} g_1 &= G_1 + (j-1)(G_2 - G_1)/J, \\ g_2 &= G_1 + j(G_2 - G_1)/J. \end{aligned} \tag{3.3}$$

The formula for the DF of the considered model is obtained by inserting formula (3.1) into (2.10). In this way one gets,

$$Q_{0N}^m(\alpha) = \sum_{j=1}^{J} v_j^m \int_{\rho_j} J_0(k\rho \sin\alpha)\rho\,d\rho. \tag{3.4}$$

If formula (3.1) is inserted into formula (2.4) then a "discrete" form of Helmholz–Rayleigh integral is obtained for both the axisymmetric as well as the far field problem.

4. Numerical calculations

A piston model was analysed to establish dependence on:
- vibration frequency,
- shape of the vibration velocity function,
- place of the vibrating surface within the baffle.

The sinus function was chosen as the vibration velocity function. Three axisymmetric sources were considered with axes:
- crossing the maximum of the sinus function (Fig. 2a)
- crossing the zero point of the sinus function (Fig. 2b)

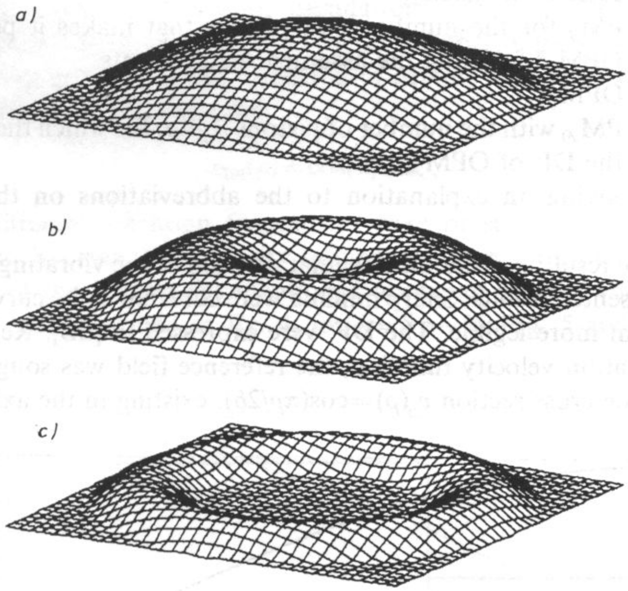


Fig. 2. The shape of the vibration velocity functions.

• outlying the sinus function (Fig. 2c), in this case a ring shaped source is obtained.

It should be noted that sinus is involved in the free vibration modes of vibrating plates and membranes. The discretization method given in this publication can be therefore applied to the model of vibrating surfaces of this kind.

The piston model with the DF fulfilling the criterion of concidence with the exact directivity was sought in two ways:

• by increasing discretization,

• for a fixed discretization the sizes of some pistons were changed.

The following symbols were introduced:
PM_J — a piston model composed of J — regular pistons,
OPM_J — an optimal piston model composed of J — irregular pistons.

While searching OPM_J numerically the boundaries between the pistons were changed with 0.01 step. Decreasing the step requires long calculations. For the practical purposes it was recognized as the OPM_J gives approximate DF (by shape and value) to the exact DF has minimal number of pistons. The difference between shorp minima of the exact DF and approximation one may be described by an angle $\Delta\alpha$ while the positions of both patterns maxima are approximately described by the values ΔD_{0N} ($\Delta\alpha \cong 0.025$ [rad] $\cong 1.43°$, $\Delta D_{0n}(\alpha) \cong 2$ [dB] were assumed).

The results of the calculations are presented in figures that present:
1 — the cross section of the vibrating surface,
2 — the plots for the following DF (or their fragments):

2.1. — calcualted exactly,

2.2. — PM_J for the number J of pistons that makes it possible to find OPM_J^a fulfilling the assumed requirements,

2.3. — OPM_J,

2.4. — PM_{J0} with the number of pistons $J_0 > J$, for which the DF is close to the DF of OPM_{J0}.

3 — a table giving an explanation to the abbreviations on the section and diagrams.

The advantage resulting from the irregular division of the vibrating surface can be seen from the presented results. Sharp minima were cut from some curves to make the remainder of them more legible. The DF were expressed in [dB]. Regardless of the shape of the vibration velocity function, the reference field was sought as constant value field with the cross section $v_0(\rho) = \cos(\pi\rho/2b)$, existing in the axis of symmetry (z-axis).

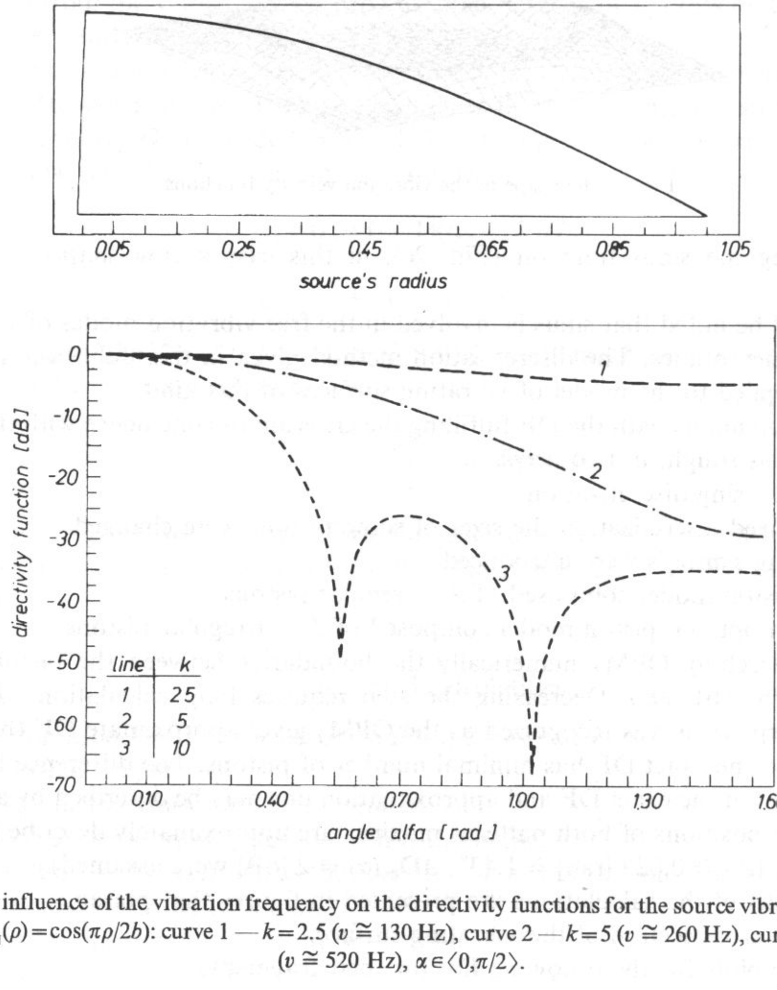


Fig. 3. The influence of the vibration frequency on the directivity functions for the source vibrating with the velocity $v_{0N}(\rho) = \cos(\pi\rho/2b)$: curve 1 — $k = 2.5$ ($v \cong 130$ Hz), curve 2 — $k = 5$ ($v \cong 260$ Hz), curve 3 — $k = 10$ ($v \cong 520$ Hz), $\alpha \in \langle 0, \pi/2 \rangle$.

In the first group of numerical calculations the influence of the vibration frequency (expressed by wave number k) on the number of pistons in the model was considered. It was assumed a fixed position of driving surface in the baffle and the following vibration velocity function

$$v_{0N}(\rho) = \cos(\pi\rho/2b), \tag{4.1}$$

The DF for different vibration frequencies were presented in Fig. 3. From this figure it follows that vibration velocity influences strongly on the shape of the DF. Then for each of the frequencies mentioned above an optimal model was sought. DF for $k=2.5$ was presented in Fg. 4, for $k=5$ — in Fig. 5 and for $k=10$ — in Fig. 6.

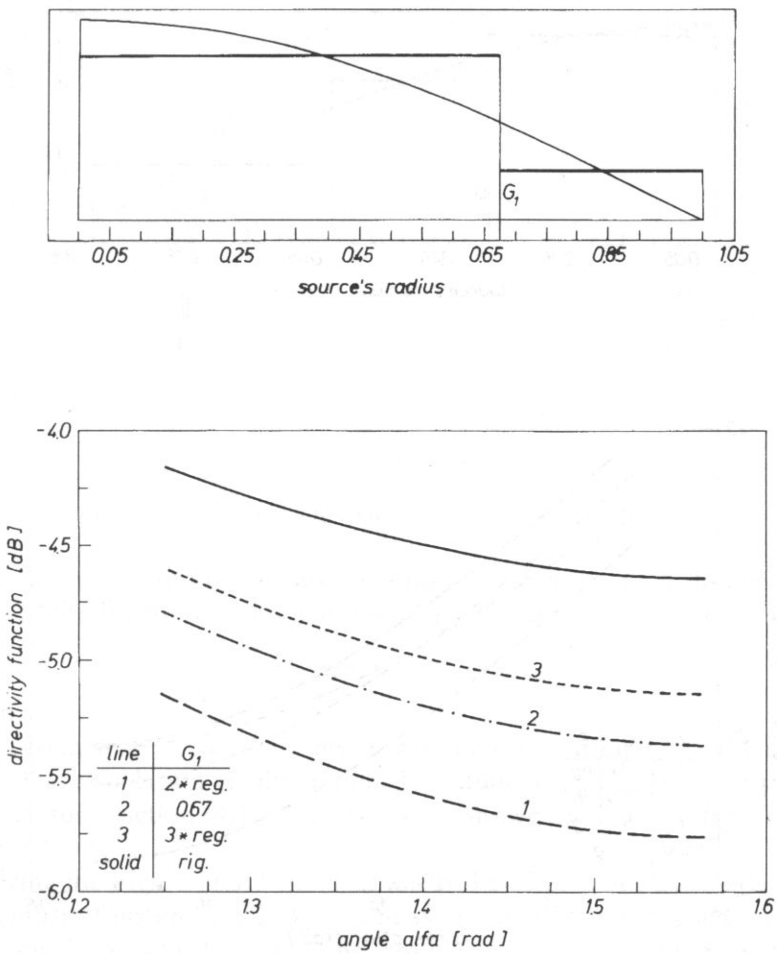


Fig. 4. The directivity functions for $k=2.5$, $v_{0N}=\cos(\pi\rho/2b)$

The figures show that the optimal model was obtained for different number of pistons depends on the frequency. For $k=2.5$ the optimal model has two pistons, for $k=5$ — three pistons, for $k=10$ — four pistons.

It has been shown that for the same number J of pistons OPM_J gives far better results than PM_J. The divergence between these models can be better observed for higher frequencies. By the model optimization it is possible to get DF quite similar to DF of PM_{J0} for less number of pistons. The number of pistons in PM_{J0} depends also on frequency.

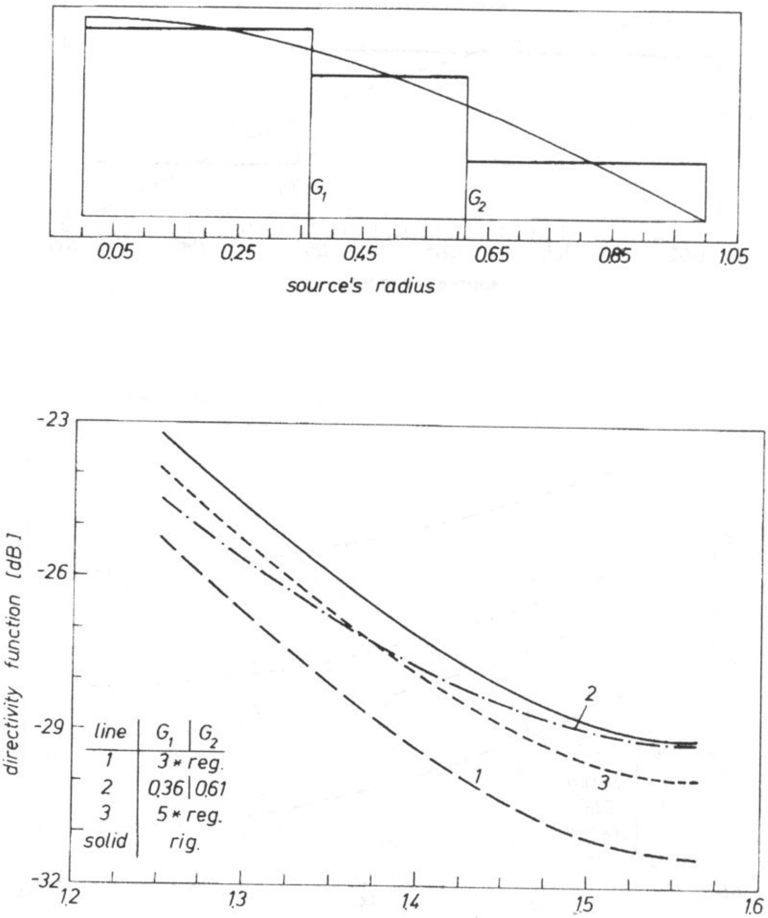


Fig. 5. The directivity functions for $k=5$, $v_{0N}=\cos(\pi\rho/2b)$.

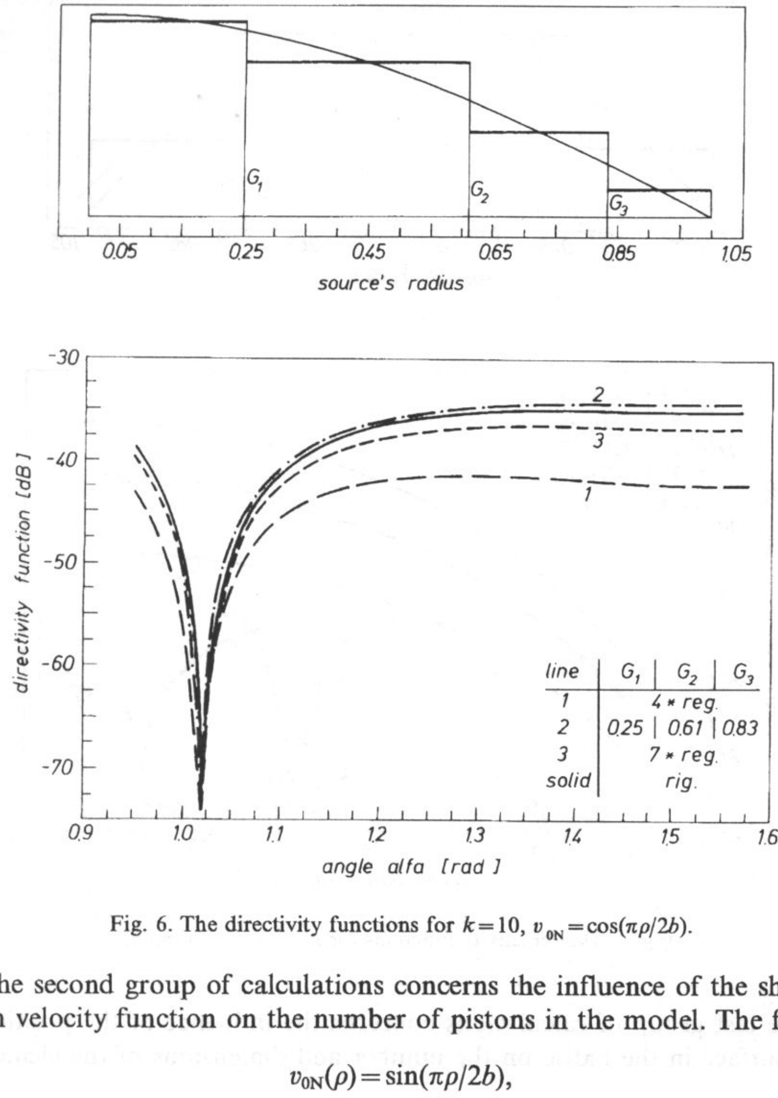


Fig. 6. The directivity functions for $k=10$, $v_{0N}=\cos(\pi\rho/2b)$.

II. The second group of calculations concerns the influence of the shape of the vibration velocity function on the number of pistons in the model. The function

$$v_{0N}(\rho)=\sin(\pi\rho/2b), \tag{4.2}$$

was chosen.

It should perhaps be noted that this is a symmetrical function with respect its maximum. The calculation results obtained for function (4.2) were compared with that obtained for function (4.1). The results are shown in Fig. 7 — for $k=5$ and in Fig. 8 — for $k=10$.

Comparing the Figs. 5 and 7 and afterwards the Figs. 6 and 8 it can be observed that the number of pistons in OPM_J as well as in PM_J does not depend on the shape of the examined function. For the same frequency, however, the dimensions of the pistons are different for different function shapes. In this case the number of pistons also depends on the vibration frequency (Fig. 7 and 8).

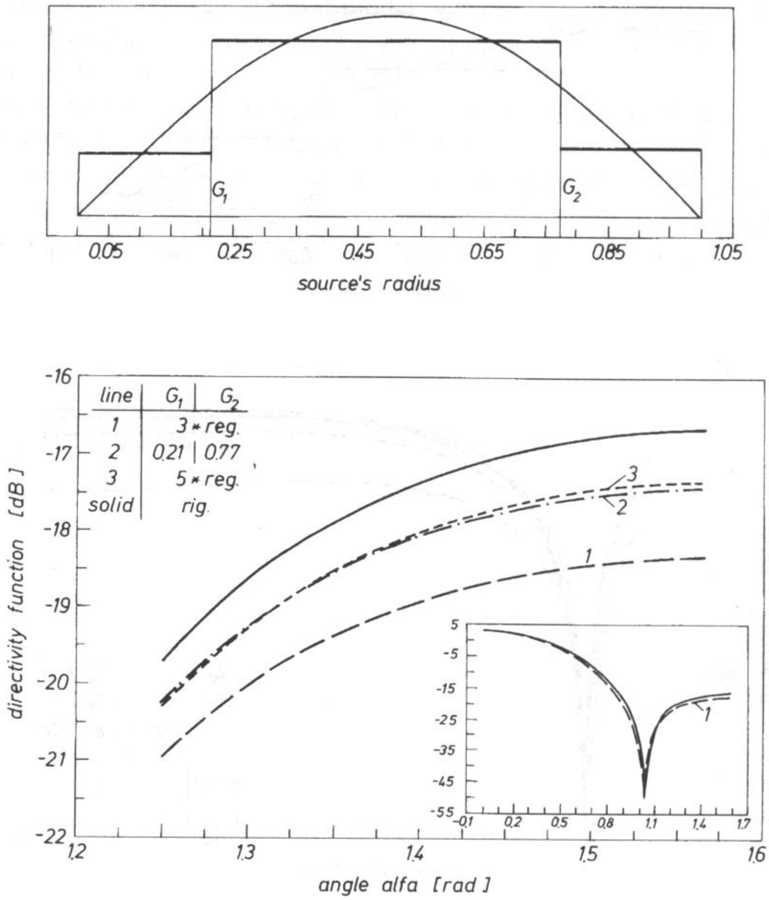


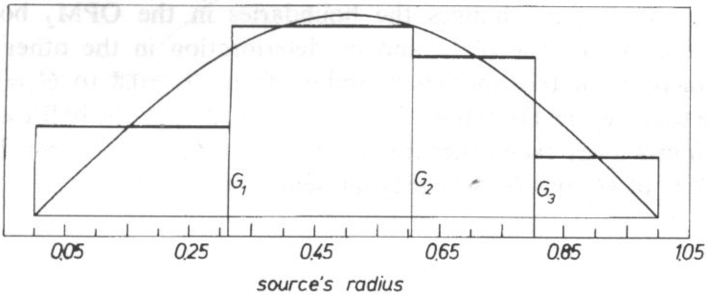
Fig. 7. The directivity functions for $k=5$, $v_{0N}=\sin(\pi\rho/b)$

III. The last group of calculations concerns the influence of the position of the vibrating surface in the baffle on the number and dimensions of the elements. The function

$$v_{0N}(\rho)=\sin(\pi(\rho-a)/(b-a)), \tag{4.3}$$

as vibration velocity was chosen. The source having such vibration velocity is a ring. Its internal and external radii were assumed to be $a=1$ and $b=2$, respectively. The calculation results were shown in Fig. 9 for $k=5$ and in Fig. 10 for $k=10$. The figures show that for the chosen position of the ring the number of pistons increases with increasing frequency.

It may be observed, comparing Figs. 7 with 9 and next with the Figs. 8 and 10, that the boundaries of the pistons in OPM_J depend on the position of the ring with respect



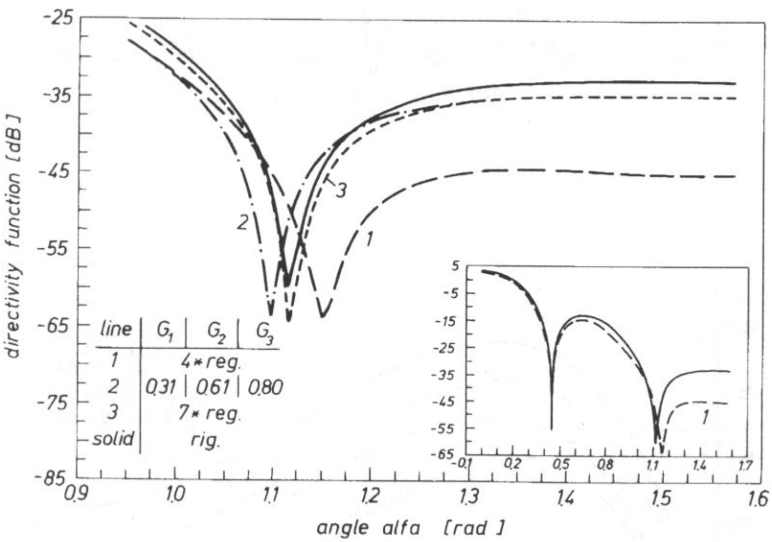


Fig. 8. The directivity functions for $k=10$, $v_{0N}=\sin(\pi\rho/b)$.

to the source axis. It turns out that the number of pistons also depends on the ring position. E.g. $OPM_{J=4}$ fulfilling the assumed requirements has not been found for $a=4$, $b=5$. Any OPM_J for $J>4$ has not been sought because of the fact that such a consideration is rather tedious.

The influence of the changes the boundaries of $OPM_{J=4}$ on the shape of DF was presented for function (4.3) for $k=10$. For fixed G_1 and G_2 boundary G_3 was changed with the step 0.01. The results are presented in Fig. 11.

That figure shows that changes the boundaries in the OPM_J both the improvement of the DF in one place and its deterioration in the other place. For example, the increase of the optimal boundary from $G_3=0.8$ to $G_3=0.81$ makes approach close together the DFs (the solid line and line 2) near the baffle at the cost of pushing the minima away each other for $G_3=0.79$, the result is inverse. The changes in the boundaries of G_1 and G_2 similarly influence on the DF.

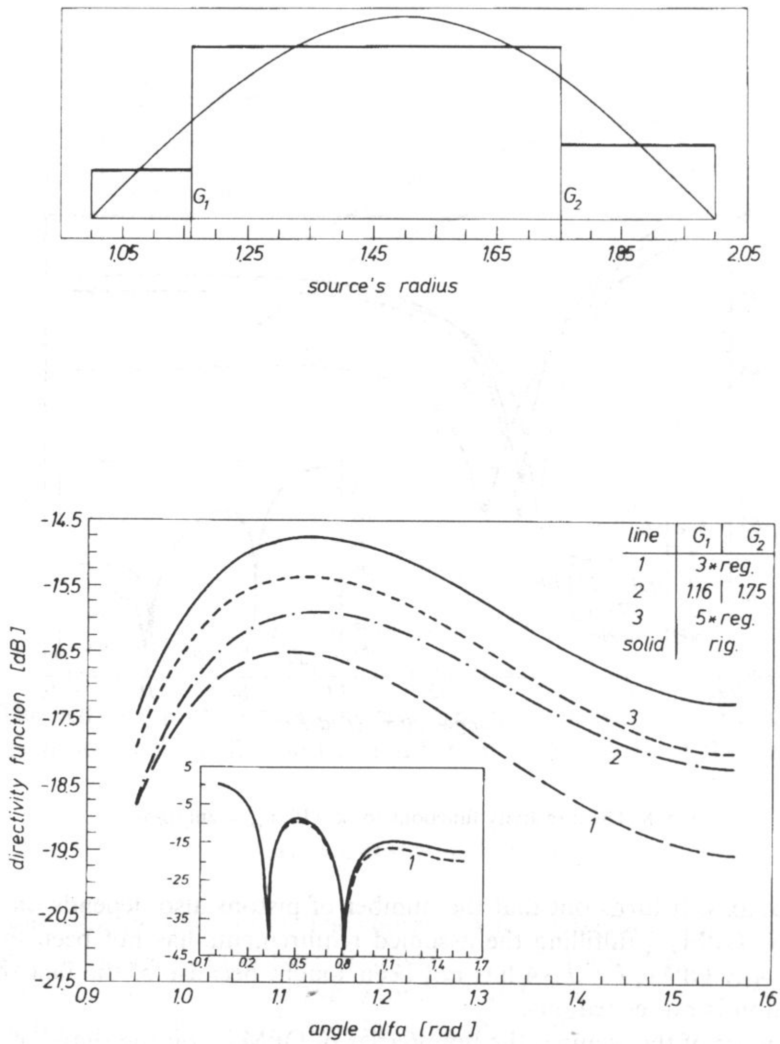


Fig. 9. The directivity functions for $k=5$, $a=1$, $b=2$, $v_{0N}(\rho)=\sin(\pi(\rho-a)/(b-a))$.

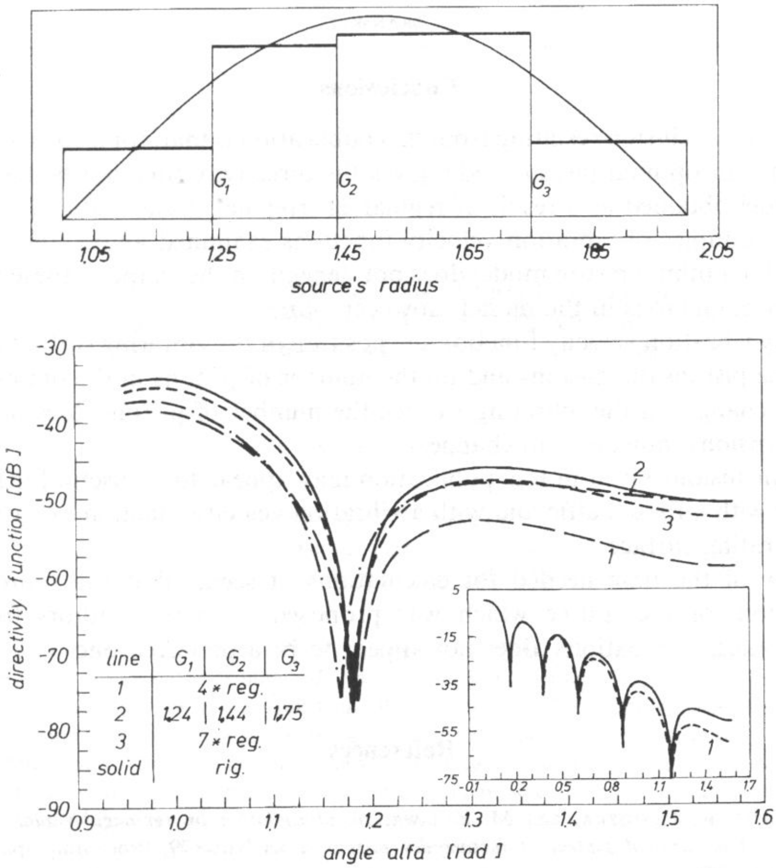


Fig. 10. The directivity functions for $k=10$, $a=1$, $b=2$, $v_{0N}=\sin(\pi(\rho-a)/(b-a))$.

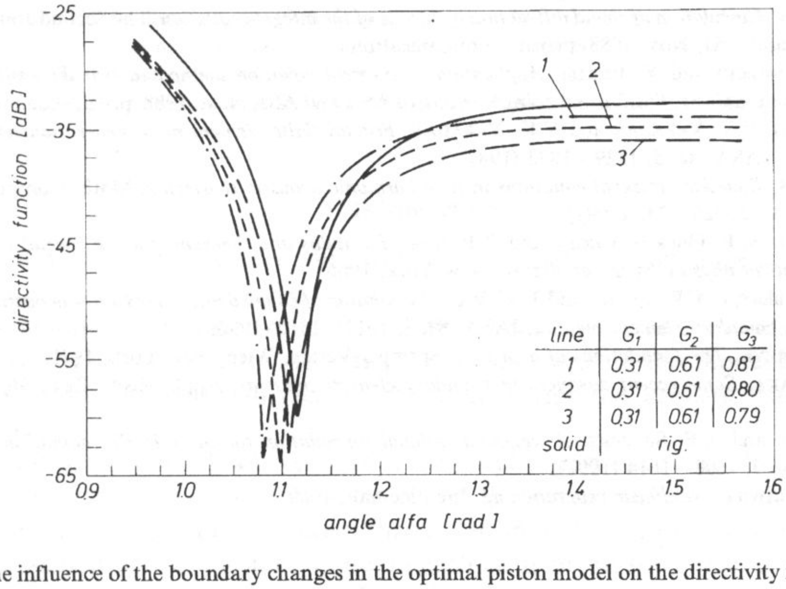


Fig. 11. The influence of the boundary changes in the optimal piston model on the directivity functions for $k=10$, $v_{0N}=\sin(\pi\rho/b)$.

Conclusions

A general conclusion resulting from this publication is that, for a constant number of elements, an optimal piston model gives the directivity function better than any piston model obtained as a result of regular or irregular discretization.

In the examples of vibration velocity functions examined above, the number of pistons in the optimal piston model does not depend on the shape of these functions. The piston boundaries in the model, however, varies.

For any vibration velocity function the position of the vibrating surface influences on both the pistons dimensions and on the number of pistons in the optimal model. For slight changes in the vibrating surface the number of pistons does not change. Their dimensions, however, do change.

The conclusions given in this publication may appear to be useful for modelling the source with a finite baffle and with a vibration velocity function of varying form on the vibrating surface.

In view of the time needed for calculations, it seems that the source regular discretization, of the source which was proposed by many authors for solving boundary integral equations, does not appear to be a preferred one.

References

[1] A. Brański, R. Panuszka, and M. Zabawa, *Iound radiation by rectangular plate represented by a model in form of system of rectangular pistons*, Inter-Noise'79, Proceedings pages 53–56, Warszawa, 1979.

[2] A. Brański, *Acoustical model of the plates* (in Polish), WSP, Rzeszów, 1991.

[3] Y. Furue, *Calculation of sound diffraction by means of the integral equation,* The Second Joint Meeting of ASA and ASJ, Nov. 1988, private communication.

[4] Y. Horinouchi and Y. Furue, *Application of integral equation method to two dimensional sound diffraction problems,* The Second Joint Meeting of ASA and ASJ, Nov., 1988, private communication.

[5] J.H. Jeng, V.V. Varadan and V.K. Varadan, *Fractal finite element mesh generation for vibration problems,* JASA, **82**, 5, 1829–1833 (1987).

[6] R. Kress, *Boundary integral equations in time-harmonic acoustic scattering*, Math. Comput. Modelling, **15**, 3–5, 229–243 (1991).

[7] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and theorems for the special functions of mathematical physics,* Springer-Verlag, New York, 1966.

[8] A.F. Seybert, C.Y.R. Cheng and T.W. Wu, *The solution of coupled interior/exterior acoustic problems using the boundary element method,* JASA, **88**, 3, 1612–1618 (1990).

[9] E. Skudrzyk, *The foundations of acoustics,* Springer-Verlag, Wien, New York, 1971.

[10] M. Tanaka, *Some recent advances in boundary element methods,* Appl. Mech. Rev., **36**, 627–634 (1983).

[11] T.W. Wu and A.F. Seybert, *A weighted residual formulation for the CHIEF method in acoustics,* JASA, **90**, 3, 1608–1614 (1991).

[12] W.I. Zangwill, *Nonlinear programming,* Prentice-hall, 1969.