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DIFFRACTION BY AN IMPEDANCE HALF-PLANE — DEPENDENCE ON THE IMPEDANCE PARAMETER

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The problem of plane wave diffraction by an impedance half-plane is considered. The aim of the paper is to analyze how the solution depends on the complex impedance parameter η. The Senior solution is analytically continued from real positive values of the parameter onto the two-sheeted Riemann surface η. It is shown that the result of the analytic continuation has two branch points of the first order and one pole. The pole is related to incidence angle of the plane wave. The proper choice of the branch is uniquely determined from the outgoing wave condition. Different types of surface waves excited on the impedance half-plane are also discussed.

Praca dotyczy problemu dyfrakcji fali płaskiej na impedancyjnej półpłaszczyźnie. Celem jej jest przeanalizowanie zależności rozwiązania od zespolonego parametru impedancyjnego η. Rozwiązanie Seniora skonstruowane dla wartości rzeczywistych parametru przedłużono analitycznie względem tego parametru. Otrzymano dwupłatową powierzchnię Riemanna η z dwoma punktami rozgałęzienia i biegunem, który jest związany z kątem padania fali płaskiej. Korzystając z warunku fali wybiegającej wyznaczono jednoznacznie gałąź, dla każdej wartości parametru należącej do płaszczyzny zmiennej zespolonej z cięciem. Przedyskutowano występowanie i zależność fali powierzchniowej od parametru.

1. Introduction

This paper is concerned with qualitative study of the solution to the problem of electromagnetic wave diffraction by an impedance half-plane. The analyzed solution is here found by using the Wiener-Hopf method and is expressed by the Fourier-type integral. The factorization is achieved by the Cauchy-type integrals. The function to be factorized depends on the impedance parameter η, and so do the factor functions. We examine the dependence of the solution on this parameter. The same solution was found by SENIOR [1] in 1952. We shall henceforth refer to the solution as the Senior solution.

The factor function is continued analytically with respect to η. A new representation of this function is obtained. It takes the form of a Cauchy-type integral along a ray in the complex plane.

The result of the continuation is that the Senior solution is analytically continued from real positive η, onto a two-sheeted Riemann surface with two branch-points and one pole. Thus, the continuation produces two possible branches of the solution. The proper branch is chosen using the outgoing wave condition. The solution is valid for both passive and active impedances. The analysis also shows the possibility of excitation of different types of surface waves on the impedance half-plane.

Formulation of the problem is given in Sections 2–3. Section 4 is concerned with mappings of the Riemann surfaces on which the functions involved are defined. Those mappings form a basis for further analysis. In Section 5 the method of analytic continuation of Cauchy-type integrals is given. In Section 6–8 the method is applied to the analytic continuation of factor function depending on two complex variables.

The extension of the Senior solution onto two-sheeted Riemann surface is described in Section 9. Different types of surface waves that emerge from the analysis are discussed in Section 10. Finally, in Appendix A index evaluation of the factorized function is given, and in Appendix B explicit formulas for the factor function are obtained.

2. Formulation of electromagnetic problem

An electromagnetic plane wave $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$ is incident upon an impendance half-plane at an angle Φ_0 to the x-axis, Fig. 1. The direction of propagation is normal

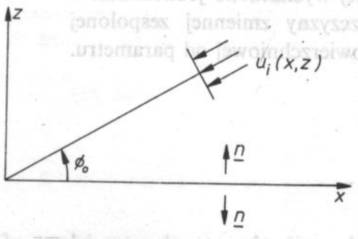


FIG. 1. Geometry of the diffraction problem

to the diffracting edge. The incident plane wave satisfies the Maxwell equations

$$\nabla\times\mathbf{H} = -i\omega\varepsilon\mathbf{E}, \qquad \nabla\times\mathbf{E} = i\omega\mu\mathbf{H}. \tag{2.1}$$

A time dependence $e^{-i\omega t}$ is assumed and suppressed throughout.

The impedance half-plane is described by the Leontovich condition

$$\mathbf{n}\times\mathbf{E} = \eta Z[\mathbf{n}\times(\mathbf{n}\times\mathbf{H})], \tag{2.2}$$

where **n** is the unit outward normal to the half-plane, $Z = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of free space, η is the reciprocal of the complex refractive index of the half-plane relative to free space.

The impedance parameter is generally a complex number. If Re $\eta \geqslant 0$ then the impedance is passive, if Re $\eta < 0$ then the impedance is active. The value $\eta = 0$ corresponds to a perfectly conducting half-plane.

The total field is a sum of the incident field and a scattered field $E^{(s)}$, $H^{(s)}$. It satisfies the Maxwell equations (2.1) in the whole space, and the Leontovich boundary condition on the half-plane. In addition the scattered field should obey the edge condition and the condition at infinity.

For the purpose of this paper those conditions are formulated in the following way:

(1) each scattered field component $E_j^{(s)}$, $H_j^{(s)}$, $j = x, y, z$, is an integrable function of ϱ (where $x = \varrho\cos\theta$, $z = \varrho\sin\theta$) in a neighbourhood of the point $\varrho = 0$, for every θ, and

(2) the scattered field contains only outgoing waves at infinity.

The above formulation of the edge condition is equivalent to the well known condition that $\int_\tau (\varepsilon|\mathbf{E}^{(s)}|^2 + \mu|\mathbf{H}^{(s)}|^2)\,d\tau =$ finite, [2], provided that each field component is of the form $\varrho^p\Phi(\theta)$, where $\Phi(\theta)$, y) is a bounded function and p is a real number, τ is a finite region of space surrounding the edge.

Splitting the field into TM (transverse magnetic) and TE (transverse electric) in the edge direction, we split the problem into two separate cases: TM, called also E polarization, where

$$\mathbf{E} = (0, E_y, 0), \quad \mathbf{H} = (H_x, 0, H_z). \tag{2.3}$$

and TE, called also H polarization, where

$$\mathbf{H} = (0, H_y, 0), \quad \mathbf{E} = (E_x, 0, E_z). \tag{2.4}$$

For each polarization the electromagnetic field (2.1) can be expressed by one scalar function u satisfying the Helmholtz equation

$$\nabla^2 u + k^2 u = 0, \tag{2.5}$$

where $k^2 = \omega^2\varepsilon\mu$.
For E polarization there is

$$E_y = u, \quad H_x = \frac{i}{\omega\mu}\frac{\partial u}{\partial z}, \quad H_z = -\frac{i}{\omega\mu}\frac{\partial u}{\partial x}. \tag{2.6}$$

For H polarization there is

$$H_y = u, \quad E_x = -\frac{i}{\omega\varepsilon}\frac{\partial u}{\partial z}, \quad E_z = \frac{i}{\omega\varepsilon}\frac{\partial u}{\partial x}. \tag{2.7}$$

From the Leontovich condition we obtain

$$u-i\,\frac{\eta}{k}\,\frac{\partial u}{\partial z}=0 \quad \text{for} \quad z=0_+, \quad x>0$$
$$u+i\,\frac{\eta}{k}\,\frac{\partial u}{\partial z}=0 \quad \text{for} \quad z=0_-, \quad x>0, \tag{2.8}$$

for E polarization, and

$$u-i\,\frac{1}{\eta k}\,\frac{\partial u}{\partial z}=0 \quad \text{for} \quad z=0_+, \quad x>0$$
$$u-i\,\frac{1}{\eta k}\,\frac{\partial u}{\partial z}=0 \quad \text{for} \quad z=0_-, \quad x>0 \tag{2.9}$$

for H polarization.

Thus, on replacing η by $\frac{1}{\eta}$ in formulas describing the field for E polarization, we obtain the field for H polarization, and vice versa.

3. *E* polarization

The incident wave is in the form

$$u_i(x,z)=e^{-ik(x\cos\Phi_0+z\sin\Phi_0)}. \tag{3.1}$$

We seek the solution of (2.5), (2.8), (3.1) as a sum of the incident wave $u_i(x, z)$ and the scattered field $u_s(x, z; \eta)=u_s$

$$u(x,z;\eta)=u_i(x,z)+u_s(x,z;\eta). \tag{3.2}$$

The scattered field satisfies the Helmholtz equation (2.5), where u_s is put in place of u, with boundary conditions for $x>0$:

$$u_s-i\,\frac{\eta}{k}\,\frac{\partial u_s}{\partial z}=-(1-\eta\sin\Phi_0)e^{ix\alpha_0} \quad \text{for} \quad z=0_+$$
$$u_s-i\,\frac{\eta}{k}\,\frac{\partial u_s}{\partial z}=-(1+\eta\sin\Phi_0)e^{ix\alpha_0} \quad \text{for} \quad z=0_-, \tag{3.3}$$

where

$$\alpha_0=-k\cos\Phi_0. \tag{3.4}$$

We seek the solution to the problem (2.5), (3.3) in the class of functions such that, via (2.6), the edge condition and condition at infinity are fulfilled.

3.1. *The method of solution*

We find the solution for a real η satisfying $0 < \eta < \infty$. We assume that $u_s(x, z; \eta)$ has the following plane wave spectral representation

$$\begin{aligned} u_s(x, z; \eta) &= \int_Q A(\alpha, \eta) e^{i\alpha x} e^{i\gamma z} d\alpha \quad \text{for} \quad z \geqslant 0, \\ u_s(x, z; \eta) &= \int_Q B(\alpha, \eta) e^{i\alpha x} e^{-i\gamma z} d\alpha \quad \text{for} \quad z \leqslant 0, \end{aligned} \tag{3.5}$$

where

$$\gamma = \sqrt{k^2 - \alpha^2}. \tag{3.6}$$

For the unique and continuous relation between α and γ, the variable α belongs to the two-sheeted Riemann surface with branch points $\alpha = \pm k$. The contour Q extends from $-\infty$ to ∞. The necessary condition of convergence of the integrals (3.5) for every x and z has the form

$$\operatorname{Im} \alpha = 0, \quad \operatorname{Im} \gamma > 0 \quad \text{as} \quad |\alpha| \to \infty. \tag{3.7}$$

The Riemann surface α is cut along the lines Γ_0, Γ_π, as in Fig. 2. The sheets are distinguished by the choice

$$\sqrt{k^2 - \alpha} = k \quad \text{for} \quad \alpha = 0 \quad \text{on sheet } \alpha_I, \tag{3.8a}$$

and

$$\sqrt{k^2 - \alpha^2} = -k \quad \text{for} \quad \alpha = 0 \quad \text{on sheet } \alpha_{II}. \tag{3.8b}$$

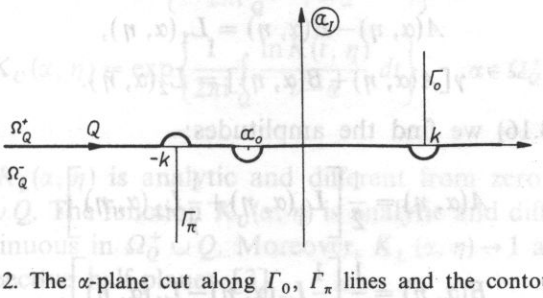


FIG. 2. The α-plane cut along Γ_0, Γ_π lines and the contour Q

The contour Q satisfying (3.7) is put along the real axis of α_I sheet, except for indentations at $\alpha = \pm k$ and $\alpha = \alpha_0$, as in. Fig. 2.

Denote by Ω_Q^+ the domain above (to the left of) the contour Q, and by Ω_Q^- the domain below (to the right of) the contour Q.

Putting (3.5) into boundary conditions (3.3) and into the condition of continuity of the total field and its normal derivative in the aperture $x < 0$, $z = 0$, we get the following integral equations for $A(\alpha, \eta)$, $B(\alpha, \eta)$:

$$\int_Q (1+\frac{\eta}{k}\gamma)\, A(\alpha, \eta) e^{i\alpha x} d\alpha = -(1-\eta \sin \Phi_0) e^{i\alpha_0 x} \quad \text{for} \quad x>0 \tag{3.9}$$

$$\int_Q (1+\frac{\eta}{k}\gamma)\, B(\alpha, \eta) e^{i\alpha x} d\alpha = -(1+\eta \sin \Phi_0) e^{i\alpha_0 x} \quad \text{for} \quad x>0 \tag{3.10}$$

$$\int_Q [A(\alpha, \eta)-B(\alpha, \eta)] e^{i\alpha x} d\alpha = 0 \quad \text{for} \quad x<0, \tag{3.11}$$

$$\int_Q \gamma [A(\alpha, \eta)+B(\alpha, \eta)] e^{i\alpha x} d\alpha = 0 \quad \text{for} \quad x<0. \tag{3.12}$$

There exists a solution of each of these equations according to the following Lemmas:

LEMMA 1. If the point α_0 lies in Ω_Q^+, there exists a solution of the equation (3.9) ((3.10)) such that

$$\left(1+\frac{\eta}{k}\gamma\right) A(\alpha, \eta) = \bar{U}_1(\alpha, \eta) - \frac{1}{2\pi i}\frac{1-\eta\sin\Phi_0}{\alpha-\alpha_0}, \tag{3.13}$$

$$\left(1+\frac{\eta}{k}\gamma\right) B(\alpha, \eta) = \bar{U}_2(\alpha, \eta) - \frac{1}{2\pi i}\frac{1+\eta\sin\Phi_0}{\alpha-\alpha_0}, \tag{3.14}$$

where $\bar{U}_1(\alpha, \eta)$ $(\bar{U}_2(\alpha, \eta))$ is an analytic function of α in the domain Ω_Q^+, and it tends to zero as $|\alpha| \to \infty$ in this domain.

LEMMA 2. Any analytic function $L_1(\alpha, \eta) (L_2(\alpha, \eta))$ in the domain Ω_Q^-, which tends to zero as $|\alpha| \to \infty$ in Ω_Q^- is a solution of the equation (3.11) ((3.12)):

$$A(\alpha, \eta) - B(\alpha, \eta) = L_1(\alpha, \eta), \tag{3.15}$$

$$\gamma [A(\alpha, \eta) + B(\alpha, \eta)] = L_2(\alpha, \eta). \tag{3.16}$$

From (3.15) and (3.16) we find the amplitudes:

$$A(\alpha, \eta) = \frac{1}{2}\left[L_1(\alpha, \eta) + \frac{1}{\gamma} L_2(\alpha, \eta)\right], \tag{3.17}$$

$$B(\alpha, \eta) = \frac{1}{2}\left[\frac{1}{\gamma} L_2(\alpha, \eta) - L_1(\alpha, \eta)\right]. \tag{3.18}$$

The functions $L_1(\alpha, \eta)$ and $L_2(\alpha, \eta)$ are to be found by solving two Wiener-Hopf equations obtained from (3.13)–(3.16):

$$U_1(\alpha, \eta) = \gamma K(\alpha, \eta) L_1(\alpha, \eta) - \frac{k \sin \Phi_0}{\pi i(\alpha-\alpha_0)}, \tag{3.19}$$

$$U_2(\alpha, \eta) = K(\alpha, \eta) L_2(\alpha, \eta) - \frac{k}{\eta \pi i(\alpha-\alpha_0)}, \tag{3.20}$$

where

$$U_1(\alpha, \eta) = \frac{k}{\eta}[\bar{U}_1(\alpha, \eta) - \bar{U}_2(\alpha, \eta)]. \tag{3.21}$$

$$U_2(\alpha, \eta) = \frac{k}{\eta}[\bar{U}_1(\alpha, \eta) + \bar{U}_2(\alpha, \eta)]. \tag{3.22}$$

$$K(\alpha, \eta) = \frac{k + \eta\sqrt{k^2 - \alpha^2}}{\eta\sqrt{k^2 - \alpha^2}}. \tag{3.23}$$

The function $K(\alpha, \eta)$ fulfils all conditions of unique factorization on Q in the class of the factors tending to 1 at infinity:

(i) $K(\alpha, \eta) \to 1$ as $|\alpha| \to \infty$,

(ii) $K(\alpha, \eta) \neq 0$ on Q, (3.24)

(iii) $\operatorname{ind}_Q K(\alpha, \eta) = \frac{1}{2\pi} \int_Q d[\arg K(\alpha, \eta)] = 0$ (the justification is in Appendix A).

Thus, in the cut α-plane there exists the unique representation

$$K(\alpha, \eta) = K_L(\alpha, \eta) K_U(\alpha, \eta), \tag{3.25}$$

where

$$K_L(\alpha, \eta) = \exp\left\{-\frac{1}{2\pi i}\int_Q \frac{\ln K(t, \eta)}{t - \alpha} dt\right\}, \quad \alpha \in \Omega_Q^-, \tag{3.26}$$

$$K_U(\alpha, \eta) = \exp\left\{\frac{1}{2\pi i}\int_Q \frac{\ln K(t, \eta)}{t - \alpha} dt\right\}, \quad \alpha \in \Omega_Q^+, \tag{3.27}$$

The function $K_L(\alpha, \eta)$ is analytic and different from zero for $\alpha \in \Omega_Q^-$, and is continuous in $\Omega_Q^- \cup Q$. The function $K_U(\alpha, \eta)$ is analytic and different from zero for $\alpha \in \Omega_Q^+$, and is continuous in $\Omega_Q^+ \cup Q$. Moreover, $K_L(\alpha, \eta) \to 1$ and $K_U(\alpha, \eta) \to 1$ as $|\alpha| \to \infty$ in the respective half-planes [3].

By the standard Wiener-Hopf method, we obtain the unique solution of the equations (3.19), (3.20) in the class of the functions tending to zero at infinity.

The solution is

$$L_1(\alpha, \eta) = \frac{k \sin \Phi_0}{\pi i(\alpha - \alpha_0)} \frac{1}{\sqrt{k + \alpha_0}} \frac{1}{\sqrt{k - \alpha}} \frac{1}{K(\alpha_0, \eta)} \frac{K_L(\alpha_0, \eta)}{K_L(\alpha, \eta)}, \tag{3.28}$$

$$L_2(\alpha, \eta) = -\frac{k}{\eta\pi i(\alpha - \alpha_0)} \frac{1}{K(\alpha_0, \eta)} \frac{K_L(\alpha_0, \eta)}{K_L(\alpha, \eta)}, \tag{3.29}$$

$$U_1(\alpha, \eta) = -\frac{k\sin\Phi_0}{\pi i(\alpha-\alpha_0)}\left[1-\sqrt{\frac{k+\alpha}{k+\alpha_0}}\frac{K_U(\alpha, \eta)}{K_U(\alpha_0, \eta)}\right], \tag{3.30}$$

$$U_2(\alpha, \eta) = -\frac{k}{\eta\pi i(\alpha-\alpha_0)}\left[1-\frac{K_U(\alpha, \eta)}{K_U(\alpha_0, \eta)}\right], \tag{3.31}$$

Putting (3.28) and (3.29) into (3.17) and (3.18) and then into (3.5) we obtain the solution to the diffraction problem:

$$u(x, z; \eta) = u_i(x, z) - \frac{k}{2\pi i}\int_Q F(\alpha, \eta)e^{i\alpha x}e^{i\gamma|z|}d\alpha, \tag{3.32}$$

where

$$F(\alpha, \eta) = \frac{1}{\alpha-\alpha_0}\frac{1}{\sqrt{k^2-\alpha^2}}\frac{1}{\eta K(\alpha_0, \eta)}\frac{K_L(\alpha_0, \eta)}{K_L(\alpha, \eta)}\left[1-\eta\frac{z}{|z|}\sqrt{\frac{k+\alpha}{k+\alpha_0}}\sin\Phi_0\right]. \tag{3.33}$$

If $\eta \to 0_+$, then (3.32) converges to Sommerfeld's solution of Dirichlet problem

$$u(x, z) = u_i(x, z) - \frac{1}{2\pi i}\int_Q \frac{1}{\alpha-\alpha_0}\sqrt{\frac{k+\alpha_0}{k+\alpha}}e^{i\alpha x}e^{i\gamma|z|}d\alpha. \tag{3.34}$$

The convergence is clearly seen if we show that there exists the limit

$$\lim_{\eta\to 0}\frac{K_L(\alpha_0, \eta)}{K_L(\alpha, \eta)} = \sqrt{\frac{k-\alpha}{k-\alpha_0}}. \tag{3.35}$$

This convergence will be verified in Section 9.

4. Conformal mappings. The location of zero-points of the function K

For every $\eta \neq 0$ the function $K(\alpha, \eta)$ given by (3.23) is defined on the two-sheeted Riemann surface α described in Section 3.

Every solution of the equation

$$k+\eta\sqrt{k^2-\alpha^2} = 0 \tag{4.1}$$

is a zero-point α_d of $K(\alpha, \eta)$. There is

$$\alpha_d = \frac{ik}{\eta}\sqrt{1-\eta^2}. \tag{4.2}$$

Thus, for a fixed $\eta \neq 0$ and $\eta \neq \pm 1$, the function $K(\alpha, \eta)$ has two zero-points on the Riemann α-surface. Both lie on either the sheet α_I or the sheet α_{II}, depending on η.

Sometimes it is convenient to use the parameter α_d instead of η. In such a case, we need one-to-one correspondence between η and α_d. The relation (4.2) expresses one-to-one correspondence (mapping) if and only if η belongs to the two-sheeted Riemann surface with branch points $\eta = \pm 1$, and α_d belongs to the two-sheeted Riemann surface with branch points $\alpha_d = \pm k$.

The mapping $\eta \rightleftarrows \alpha_d$ is shown in Fig. 3 abcd. Each surface is divided into sheets, and every sheet is shown separately. The division is arbitrary, but here it is done in the way suitable for further investigations. The sheets η_I and η_{II} are chosen in such a way that the passage from one sheet to the other is through the branch cut along the interval $[-1, 1]$ of the real axis, Fig. 3 ab.

The sheets η_{I} and η_{II} are distinguished by the choice

$$\alpha_d^{(1)} = k\sqrt{2} \quad \text{for } \eta = i \quad \text{on } \eta_I, \text{ and} \tag{4.3a}$$

$$\alpha_d^{(2)} = -k\sqrt{2} \quad \text{for } \eta = i \quad \text{on } \eta_{II}, \tag{4.3b}$$

By the choice of the number (4.3a) we define the branch of the function (4.2) which we also will denote by $\alpha_d^{(1)}$. The second branch we will denote by $\alpha_d^{(2)}$.

There is

$$\alpha_d^{(2)} = -\alpha_d^{(1)} \quad \text{for every } \eta \notin [-1, 1], \tag{4.4}$$

and

$$\operatorname{Re}\alpha_d^{(1)} \geqq 0, \qquad \operatorname{Re}\alpha^{(2)d} \geqq 0 \tag{4.5}$$

Now we present the Riemann surface α_d divided into sheets as in Fig. 3 cd.

The sheets α_{dI} and α_{dII} are chosen in the same way as the sheets α_I and α_{II} described by (3.8). The passage from one sheet to the other is through branch cut lines Γ_0, Γ_π.

By the function (4.2) with chosen branches $\alpha_d^{(1)}$, $\alpha_d^{(2)}$, sheet η_I, is mapped onto Re $\alpha_{dI} \geqslant 0$, Re $\alpha_{dII} \geqslant 0$ in Fig. 3 cd, and sheet η_{II} is mapped onto Re $\alpha_{dI} \leqslant 0$, Re $\alpha_{dII} \leqslant 0$. The corresponding domains are denoted by the numbers 1–6 and 7–12 respectively.

From (4.1) we have the relation

$$\gamma_d = -\frac{k}{\eta}, \tag{4.6}$$

where $\gamma_d = \sqrt{k^2 - \alpha_d^2}$.

The mappings $\eta \rightleftarrows \gamma_d$ and $\alpha_d \rightleftarrows \gamma_d$ are shown in Fig. 3 abef and Fig. 3 cdef respectively.

The function

$$\beta_d = \cos^{-1}\left(\frac{\alpha_d}{k}\right) \tag{4.7a}$$

defines one-to-one mapping of the two-sheeted Riemann surface α_d onto a cylinder

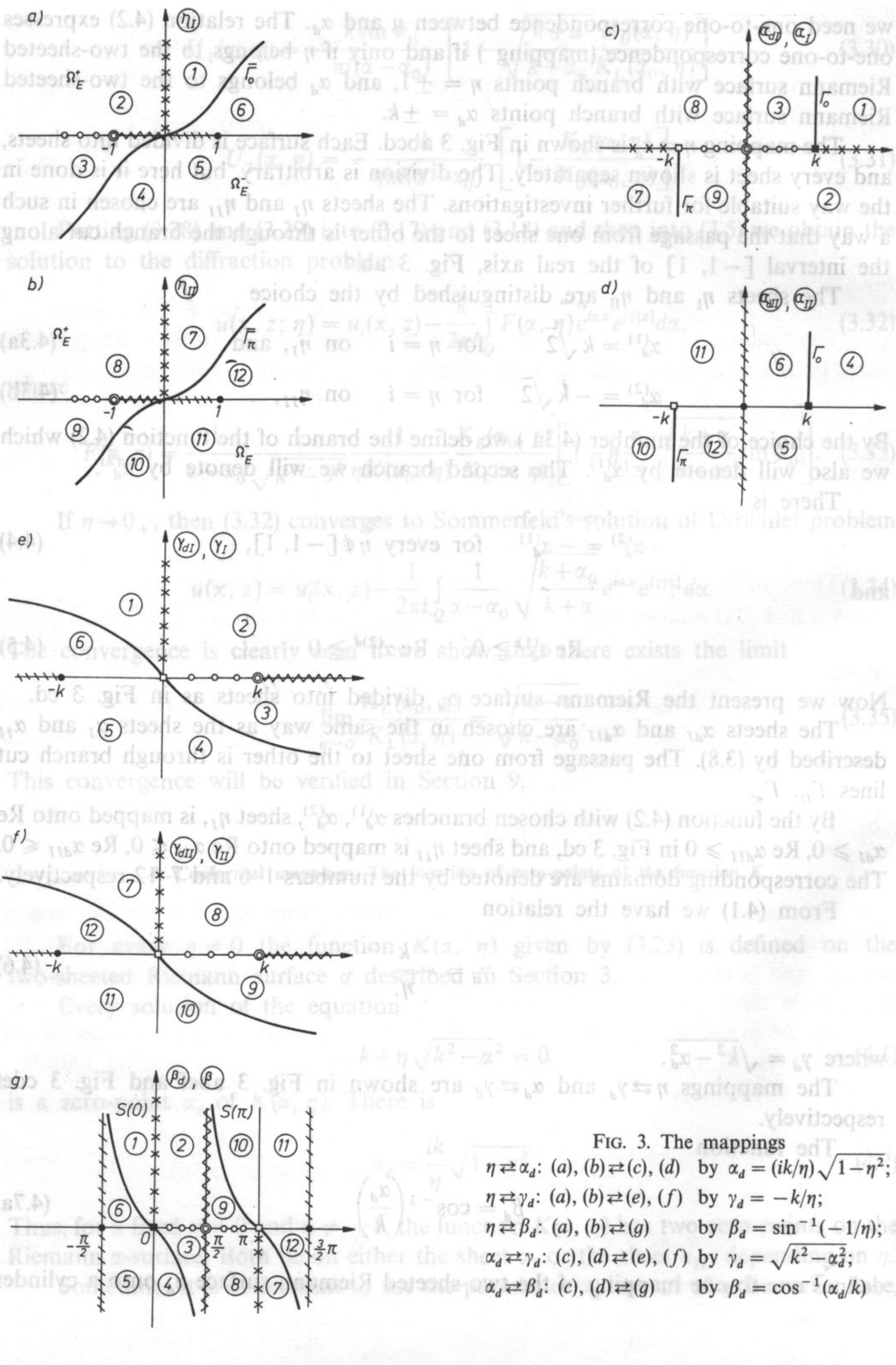


FIG. 3. The mappings

$\eta \rightleftarrows \alpha_d$: (a), (b) $\rightleftarrows$ (c), (d) by $\alpha_d = (ik/\eta)\sqrt{1-\eta^2}$;

$\eta \rightleftarrows \gamma_d$: (a), (b) $\rightleftarrows$ (e), (f) by $\gamma_d = -k/\eta$;

$\eta \rightleftarrows \beta_d$: (a), (b) $\rightleftarrows$ (g) by $\beta_d = \sin^{-1}(-1/\eta)$;

$\alpha_d \rightleftarrows \gamma_d$: (c), (d) $\rightleftarrows$ (e), (f) by $\gamma_d = \sqrt{k^2-\alpha_d^2}$;

$\alpha_d \rightleftarrows \beta_d$: (c), (d) $\rightleftarrows$ (g) by $\beta_d = \cos^{-1}(\alpha_d/k)$

surface β_d, as is shown in Fig. 3 cdg. In Fig. 3 g the cylinder is cut along a generator denoted twice: Re $\beta_d = -\frac{\pi}{2}$, and Re $\beta_d = \frac{3}{2}\pi$.

The cut cylinder can be also unfolded to the plane

$$\beta = \cos^{-1}\left(\frac{\alpha}{k}\right) \tag{4.7b}$$

which will be used in asymptotic evaluation of the solution (3.32). The lines $S(0)$ and $S(\Pi)$ are the steepest descent paths for $\theta = 0$ and $\theta = \Pi$ respectively. In the transformation (4.7) Γ_0 is mapped onto $S(0)$ and $\Gamma(\Pi)$ onto $S(\Pi)$. The sheet α_{dI}, (α_1) is mapped onto the domain contained between $S(0)$ and $S(\Pi)$.

The function

$$\beta_d = \sin^{-1}\left(-\frac{1}{\eta}\right) \tag{4.8}$$

defines one-to-one mapping of the Riemann surface η onto the cylinder β_d, Fig. 3 abg.

The mappings contain valuable information needed for the analysis of the solution as a function of η. Some of them we mention here:

1. Let us treat η as the plane of the complex parameter. If we put sheet η_{I} on sheet η_{II} the curves $\bar{\Gamma}_0$ and $\bar{\Gamma}_\pi$ cover each other. We denote the resulting single curve by S_E; the domain above S_E by Ω_E^+; and the domain below S_E by Ω_E^- respectively. These domains have the following properties:

(i) For $\eta \in \Omega_E^+$ the zero-points $\alpha_d^{(1)}$, $\alpha_d^{(2)}$ lie on the sheet α_I.

(ii) For $\eta \in \Omega_E^-$ the zero-points $\alpha_d^{(1)}$, $\alpha_d^{(2)}$ lie on the sheet α_{II}.

2. Let the function $K(\alpha, \eta)$ be factorized on a contour Q fulfilling (3.7). We denote by $L(Q)$ the set of the points on the η-plane for which the zero points $\alpha_d^{(1)}$, $\alpha_d^{(2)}$ lie on Q. Then we can check the following facts:

(i) For Q put as in Fig. 2 (dotted and crossed real axis in Fig. 3c), we have $L(Q)$ consisting of imaginary semiaxis and a part of negative real semiaxis (dotted and

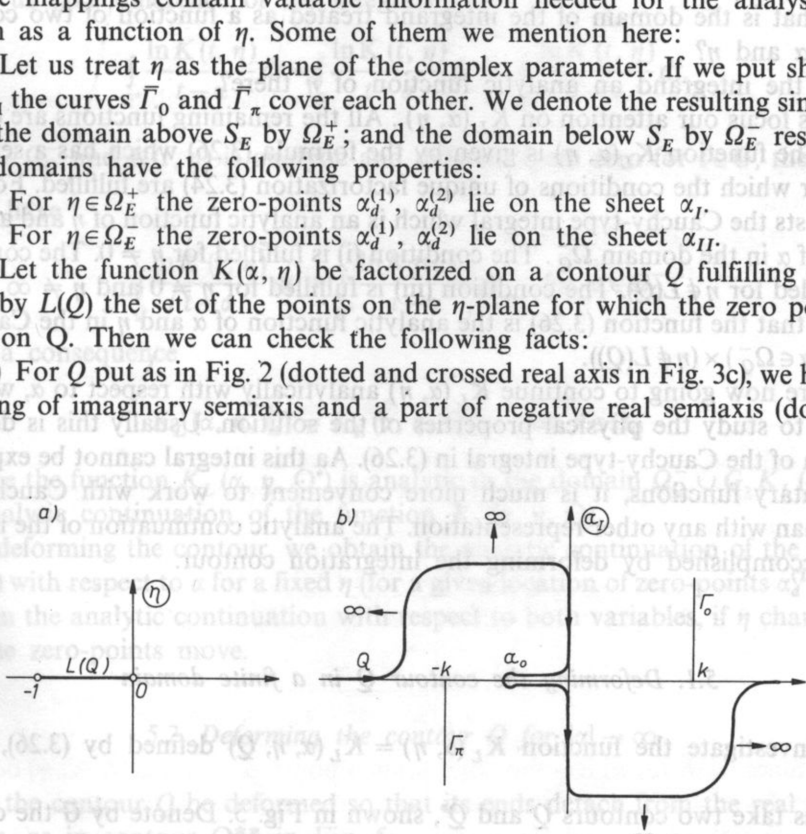


FIG. 4. The η and α-planes, (a) the Line $L(Q)$, and (b) corresponding to $L(Q)$ location of the contour Q

crossed line in Fig. 3 ab)

$$L(Q) = \{\eta: (\mathrm{Re}\eta = 0,\ \mathrm{Im}\eta \geqq 0) \cup (\mathrm{Re}\eta \leqq -1,\ \mathrm{Im}\eta = 0)\}. \tag{4.9}$$

(ii) Every contour Q which passes through $\alpha_I = 0$ and $\alpha_I = \infty$ is mapped onto $L(Q)$ with endpoints $\eta = -1$ and $\eta = 0$.

(iii) For $L(Q)$ which joins the points $\eta = -1$ and $\eta = 0$ in the simplest way, as in Fig. 4a, the contour Q is put along the imaginary axis of α_I. To fulfil the condition (3.7) it has the ends put on the real axis, Fig. 4b. This location of Q will be used in Section 8 and 9.

5. The method of analytic continuation of the factor function

In the following analysis we will treat the parameter η as a variable. Let us return to the solution (3.32) and answer the following questions:

1. For which η does the formula (3.32) have sense?
2. What is the domain of the integrand treated as a function of two complex variables α and η?
3. Is the integrand an analytic function of η there?

Let us focus our attention on $K_L(\alpha, \eta)$. All the remaining functions are easy to analyze. The function $K_L(\alpha, \eta)$ is given by the formula (3.26) which has a sense for every η for which the conditions of unique factorization (3.24) are fulfilled. For such η there exists the Cauchy-type integral which is an analytic function of η and analytic function of α in the domain Ω_Q^-. The condition (i) is fulfilled for $\eta \neq 0$. The condition (ii) is fulfilled for $\eta \notin L(Q)$. The condition (iii) is fulfilled for $\eta \neq 0$ and $\eta \neq \infty$. So we can claim that the function (3.26) is the analytic function of α and η in the Cartesian product $(\alpha \in \Omega_Q^-) \times (\eta \notin L(Q))$.

We are now going to continue $K_L(\alpha, \eta)$ analytically with respect to α, which is necessary to study the physical properties of the solution. Usually this is done by evaluation of the Cauchy-type integral in (3.26). Aa this integral cannot be expressed by elementary functions, it is much more convenient to work with Cauchy-type integral than with any other representation. The analytic continuation of the integral will be accomplished by deforming the integration contour.

5.1. *Deforming the contour Q in a finite domain*

We investigate the function $K_L(\alpha, \eta) = K_L(\alpha, \eta, Q)$ defined by (3.26), where $\eta \notin L(Q)$.

Let us take two contours Q and Q^*, shown in Fig. 5. Denote by G the domain contained between the contours, and by δG its boundary.

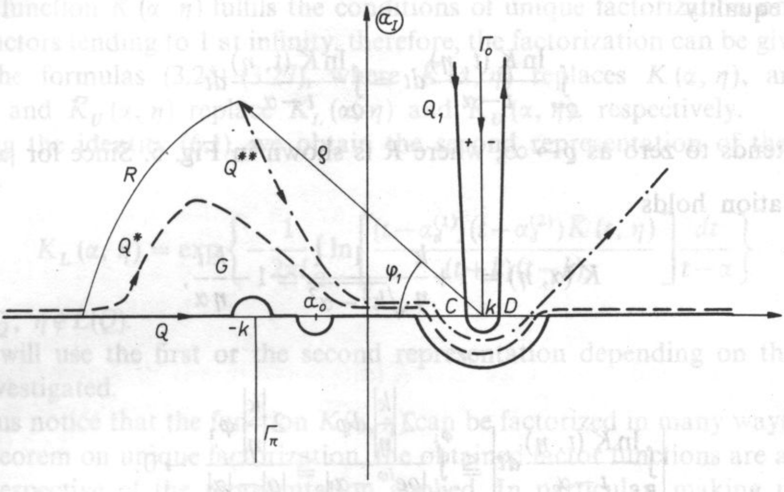


FIG. 5. The contour Q and its deformations for analytic continuation of the factor function

Let us take difference of the integrals

$$\int_{Q^*} \frac{\ln K(t,\eta)}{t-\alpha}\,dt - \int_{Q} \frac{\ln K(t,\eta)}{t-\alpha}\,dt = \oint_{\delta G} \frac{\ln K(t,\eta)}{t-\alpha}\,dt. \tag{5.1}$$

If $\alpha \in \Omega_Q^-$ and $K(t, \eta)$ is analytic and different from zero for $t \in G$, then $\oint_{\delta G} = 0$ and we have

$$\int_{Q} \frac{\ln K(t,\eta)}{t-\alpha}\,dt = \int_{Q^*} \frac{\ln K(t,\eta)}{t-\alpha}\,dt \qquad \text{for } \alpha \in \Omega_Q^-. \tag{5.2}$$

As a consequence

$$K_L(\alpha, n, Q) = K_L(\alpha, \eta, Q^*) \qquad \text{for } \alpha \in \Omega_Q^-. \tag{5.3}$$

Since the function $K_L(\alpha, \eta, Q^*)$ is analytic in the domain $\Omega_Q^- \cup G$, $K_L(\alpha, \eta, Q^*)$ is an analytic continuation of the function $K_L(\alpha, \eta, Q)$.

By deforming the contour, we obtain the analytic continuation of the function $K_L(\alpha, \eta)$ with respect to α for a fixed η (for a given location of zero-points $\alpha_d^{(1)}, \alpha_d^{(2)}$) or we obtain the analytic continuation with respect to both variables, if η changes and hence the zero-points move.

5.2. *Deforming the contour Q for $|\alpha| \to \infty$*

Let the contour Q be deformed so that its ends detach from the real axis and move up, as in contour Q** in Fig. 5.

The equality

$$\int_{Q^{**}} \frac{\ln K(t, \eta)}{t-\alpha} dt = \int_{Q} \frac{\ln K(t, \eta)}{t-\alpha} dt \tag{5.4}$$

holds if $\int_R$ tends to zero as $\varrho \to \infty$, where R is shown in Fig. 5. Since for $|\alpha| \to \infty$ the approximation holds

$$K(\alpha, \eta) = 1 + \frac{k}{\eta} \frac{1}{\sqrt{k^2-\alpha^2}} \simeq 1 - \frac{k}{\eta} \frac{i}{\alpha}, \tag{5.5}$$

therefore

$$\left| \int_R \frac{\ln K(t, \eta)}{t-\alpha} dt \right| \leqq \int_0^{\varphi_1} \frac{2\left|\frac{k}{\eta}\right| d\varphi}{|\varrho e^{i\varphi} - \alpha|} \leqq \frac{2\left|\frac{k}{\eta}\right| \varphi_1}{|\varrho| - |\alpha|} \underset{\substack{\varrho \to \infty \\ \eta \neq 0}}{\to} 0. \tag{5.6}$$

As a result, we have the equality

$$K_L(\alpha, \eta, Q) = K_L(\alpha, \eta, Q^{**}) \qquad \text{for } \alpha \in \Omega_Q^- . \tag{5.7}$$

Corollary. The function $K_L(\alpha, \eta)$ can be continued analytically by deforming the contour Q towards Ω_Q^+ into a domain which contains neither a branch-point nor a zero-point of the function $K(\alpha, \eta)$. The ends of the contour Q can be moved above the real axis.

For $\eta \in \Omega_E^-$ we can deform the contour Q to the contour Q_1, Fig. 5. For $\eta \in \Omega_E^+$ the zero-point $\alpha_d^{(1)}$ does not allow us to shift Q so far. In this case we will use different representation of $K_L(\alpha, \eta)$.

6. The alternative representation of the factor function

Let us introduce the function $\bar{K}(\alpha, \eta)$ defined by the following identity with respect to α and η:

$$\bar{K}(\alpha, \eta) \equiv \frac{K(\alpha, \eta)(\alpha+k)(\alpha-k)}{(\alpha-\alpha_d^{(1)})(\alpha-\alpha_d^{(2)})}, \tag{6.1a}$$

$$\bar{K}(\alpha, \eta) = \frac{\eta\sqrt{k^2-\alpha^2}}{\eta\sqrt{k^2-\alpha^2}-k}. \tag{6.1b}$$

The function $\bar{K}(\alpha, \eta)$ has the same branch points $\alpha = \pm k$ as $K(\alpha, \eta)$ does, and has two poles $\alpha = \alpha_d^{(1)}$, $\alpha = \alpha_d^{(2)}$, which unlike the zeros of $K(\alpha, \eta)$ are located for $\eta \in \Omega_E^-$ on the sheet α_I, and for $\eta \in \Omega_E^+$ on the sheet α_{II}.

The function $\bar{K}(\alpha, \eta)$ fulfils the conditions of unique factorization on Q in the class of factors tending to 1 at infinity, therefore, the factorization can be given by the use of the formulas (3.25)–(3.27), where $\bar{K}(\alpha, \eta)$ replaces $K(\alpha, \eta)$, and where $\bar{K}_L(\alpha, \eta)$ and $\bar{K}_U(\alpha, \eta)$ replace $K_L(\alpha, \eta)$ and $K_U(\alpha, \eta)$, respectively.

Using the identity (6.1), we obtain the second representation of the function $K_L(\alpha, \eta)$.

$$K_L(\alpha, \eta) = \exp\left\{-\frac{1}{2\pi i}\int_Q \ln\left[\frac{(t-\alpha_d^{(1)})(t-\alpha_d^{(2)})\bar{K}(t, \eta)}{(t+k)(t-k)}\right]\frac{dt}{t-\alpha}\right\} \tag{6.2}$$

for $\alpha \in \Omega_Q^-$, $\eta \notin L(Q)$.

We will use the first or the second representation depending on the domain being investigated.

Let us notice that the function $K(\alpha, \eta)$ can be factorized in many ways, and yet, by the theorem on unique factorization, the obtained factor functions are always the same, irrespective of the representation applied. In particular, making use of the identity (6.1a) and the properties of the factor functions, we can write

$$K_L(\alpha, \eta) = \frac{\alpha-\alpha_d^{(1)}}{\alpha-k}\bar{K}_L(\alpha, \eta) = \frac{\alpha-\alpha_d^{(1)}}{\alpha-k}\exp\left\{\frac{1}{2\pi i}\int_Q \frac{\ln\bar{K}(t, \eta)}{t-\alpha}dt\right\} \tag{6.3}$$

for $\eta \notin L(Q)$, $\alpha \in \Omega_Q^-$, where $\alpha_d^{(1)} \in \Omega_Q^+$.

$$K_U(\alpha, \eta) = \frac{\alpha-\alpha_d^{(2)}}{\alpha+k}\bar{K}_U(\alpha, \eta) = \frac{\alpha-\alpha_d^{(2)}}{\alpha+k}\exp\left\{-\frac{1}{2\pi i}\int_Q \frac{\ln\bar{K}(t, \eta)}{t-\alpha}dt\right\} \tag{6.4}$$

for $\eta \notin L(Q)$, $\alpha \in \Omega_Q^+$, where $\alpha_d^{(2)} \in \Omega_Q^-$.

The functions (6.3) and (3.26) are equal as well as the functions (6.4) and (3.27) are.

7. Analytic continuation of $K_L(\alpha, \eta)$ onto the Cartesian product $(\alpha \notin \Gamma_0) \times (\eta \notin L(Q))$

Theorem 1. *There exists the analytic continuation of the factor function* $K_L(\alpha, \eta)$ *given by* (3.26) *onto the Cartesian product of cut α-plane and cut η-plane. For* $\alpha \notin \Gamma_0$ *this analytic continuation has the representation in the form*

$$K_L(\alpha, \eta) = \exp\left\{\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta)}{t-\alpha}dt\right\} \tag{7.1a}$$

for $\eta \in \Omega_E^-$, and

$$K_L(\alpha, \eta) = \frac{\alpha-\alpha_d^{(1)}}{\alpha-k}\exp\left\{\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta)-2\pi i}{t-\alpha}dt\right\} \tag{7.1b}$$

for $(\eta \in \Omega_E^+) \cap (\eta \notin L(Q))$, where

$$K^*(\alpha, \eta) = \frac{\eta\sqrt{k^2-\alpha^2}+k}{\eta\sqrt{k^2-\alpha^2}-k}, \tag{7.2}$$

Im $(\ln K^*) \in (-\pi, \pi$, for $\eta \in \Omega_E^-$ the line Γ_0 is directed as in Fig. 5, $\alpha_d^{(1)} \in \Omega_Q^+$.

Proof. Let $\eta \in \Omega_E^-$. According to the Corollary in Section 5, for $\eta \in \Omega_E^-$, $\alpha \in \Omega_Q^-$ we have the equality

$$\int_Q \frac{\ln K(t, \eta)}{t-\alpha} dt = \int_{Q_1} \frac{\ln K(t, \eta)}{t-\alpha} dt. \tag{7.3}$$

In the neighbourhood of $\alpha = k$ we have $K(\alpha, \eta) = O[(k-\alpha)^{-1/2}]$ and, therefore, the integral on a half-circle CD, Fig. 5, tends to zero as its radius tends to zero. Thus, we have

$$\int_Q \frac{\ln K(t, \eta)}{t-\alpha} dt = \int_{\Gamma_0} \ln \left[\frac{K_+(t, \eta)}{K_-(t, \eta)}\right] \frac{dt}{t-\alpha}. \tag{7.4}$$

The subscripts "+" and "−" denote the values of the function $K(t, \eta)$ from the right hand side and left hand side of Γ_0, respectively. According to (3.8), we have

$$(\sqrt{k^2-t^2})_+ = \sqrt{k^2-t^2}, \quad (\sqrt{k^2-t^2})_- = -\sqrt{k^2-t^2}, \tag{7.5}$$

and, therefore,

$$\frac{K_+(t, \eta)}{K_-(t, \eta)} = K^*(t, \eta). \tag{7.6}$$

Substituting (7.6) into (7.4), and then into (3.26) we obtain (7.1a).

Let $\eta \in \Omega_E^+$. We use the representation (6.3). In exactly the same way as above, we obtain the equality:

$$\int_Q \frac{\ln \bar{K}(t, \eta)}{t-\alpha} dt = \int_{\Gamma_0} \ln \left[\frac{\bar{K}_+(t, \eta)}{\bar{K}_-(t, \eta)}\right] \frac{dt}{t-\alpha}. \tag{7.7}$$

From (6.1b) and (7.5), we have

$$\frac{\bar{K}_+(t, \eta)}{\bar{K}_-(t, \eta)} = K^*(t, \eta). \tag{7.8}$$

Substituting (7.8) into (7.7) and then into (6.3) we obtain (7.1b).

The branch of logarithm is chosen such that the function $K_L(\alpha, \eta)$ is as singular at the point $\alpha = k$, as the function $K(\alpha, \eta)$ is:

$$K_L(\alpha, \eta) = O[(\alpha-k)^{-1/2}] \text{ at } \alpha = k. \tag{7.9}$$

Then the integrals in (7.1) are convergent.

The values of $K_L(\alpha, \eta)$ for $\eta \in S_E$ can be obtained as a limit in the formula (7.1a) or (7.1b) as $\eta \to S_E$

8. Analytic continuation of $K_L(\alpha, \eta)$ onto the second sheet of the Riemann surface η

Theorem 2. *For fixed α the function $K_L(\alpha, \eta)$* has two branch points $\eta = -1$, $\eta = 0$ *of the first order.*

Proof. As far as the point $\eta = -1$ is concerned the conclusions follows immediately from (7.1b). Since the exponential function is regular at $\eta = -1$, the function $K(\alpha, \eta)$ is as singular as $\alpha_d^{(1)}$ is and, therefore it behaves like $\sqrt{1-\eta^2}$. This is illustrated in Fig. 6 a, b. As η circulates twice round the point $\eta = -1$ along a closed line C_1 on the plane η (Fig. 6a), the zero point α_d circulates once round the point $\alpha_d = 0$ along the line $\bar{C}_1$ (Fig. 6b), and returns to the initial position, so that the value $K_L(\alpha, \eta)$ returns to the initial value.

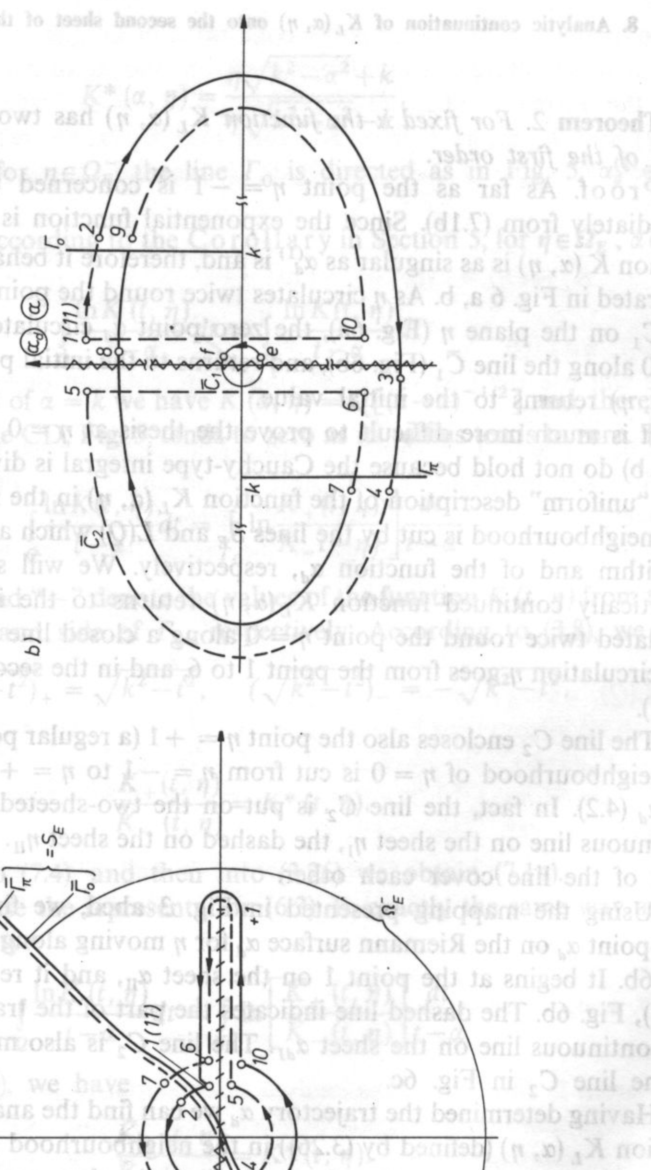
It is much more difficult to prove the thesis at $\eta = 0$. For $\eta = 0$ the formulas (7.1a, b) do not hold because the Cauchy-type integral is divergent. Moreover, there is no "uniform" description of the function $K_L(\alpha, \eta)$ in the neighbourhood of $\eta = 0$. This neighbourhood is cut by the lines S_E and $L(Q)$ which are the branch cuts of the logarithm and of the function α_d, respectively. We will show, however, that the analytically continued function $K_L(\alpha, \eta)$ returns to the initial value after η has circulated twice round the point $\eta = 0$ along a closed line C_2, as in Fig. 6a. In the first circulation η goes from the point 1 to 6, and in the second circulation from 6 to 1 (11).

The line C_2 encloses also the point $\eta = +1$ (a regular point of $K_L(\alpha, \eta)$, because the neighbourhood of $\eta = 0$ is cut from $\eta = -1$ to $\eta = +1$ by the transformation $\eta \rightleftarrows \alpha_d$ (4.2). In fact, the line C_2 is put on the two-sheeted Riemann surface η; the continuous line on the sheet η_{I}, the dashed on the sheet η_{II}. In the η–plane these two parts of the line cover each other.

Using the mapping presented in Fig. 3 abcd, we find the trajectory of the zero-point α_d on the Riemann surface α_d for η moving along C_2. This is the line $\bar{C}_2$ in Fig. 6b. It begins at the point 1 on the sheet α_{II}, and it returns to the initial point 1 (11), Fig. 6b. The dashed line indicates the part of the trajectory on the sheet $\alpha_{d\text{II}}$, the continuous line on the sheet α_{dI}. The line C_2 is also mapped into the β_d–plane – the line C_2 in Fig. 6c.

Having determined the trajectory α_d we can find the analytic continuation of the function $K_L(\alpha, \eta)$ (defined by (3.26)) in the neighbourhood of $\eta = 0$. Three contours selected from eleven ones $Q = Q_i$, $i = 1, 2, \ldots, 11$, that result from pushing Q by the moving point α_d are shown in Fig. 7. The contour Q_i corresponds to the location α_d in the point $i = 1, 2, \ldots, 11$, as in Fig. 6b. This is the realization of analytic continuation of the Cauchy–type integral on Q. The $\alpha_d^{(1)}$, $\alpha_d^{(2)}$ points are additional, dependent on η branch points of the logarithm in this integral.

Looking at the trajectory of α_d, Fig. 6b, we can draw the contour Q_{11}. Although it is in the form of a very long loop it can be shown that the Cauchy–type integral on Q_{11} is equal to the Cauchy–type integral on Q_1, since the integral along a part of the contour "pushed" by $\alpha_d^{(1)}$in first circulation is reduced by the integral along the same



Theorem 2. *For fixed* $\varkappa$ *the function* $K_1(\alpha, \eta)$ *has two branch points* $\eta = -1$, $\eta = 0$ *of the first order.*

Proof. As far as the point $\eta = -1$ is concerned the conclusions follows immediately from (7.1b). Since the exponential function is regular at $\eta = -1$, the function $K(\alpha, \eta)$ is as singular as $\alpha_d^{(1)}$ is and, therefore it behaves like $\sqrt{1-\eta}$. This is illustrated in Fig. 6 a, b. As η circulates twice round the point $\eta = -1$ along a closed line C_1 on the plane η (Fig. 6a), the zero-point α_d circulates once round the point $\alpha_d = 0$ along the line $\bar{C}_1$ (Fig. 6b), and comes to the initial position, so that the value $K_1(\alpha, \eta)$ returns to the initial value.

It is much more difficult to prove the thesis at $\eta = 0$. For $\eta = 0$ the formulas (7.1a, b) do not hold because the Cauchy-type integral is divergent. Moreover, there is no "uniform" description of the function $K_1(\alpha, \eta)$ in the neighbourhood of $\eta = 0$. This neighbourhood is cut by the lines $\bar{S}_E$ and $L(Q)$ which are the branch cuts of the logarithm and of the function α_d, respectively. We will show, however, that the analytically continued function $K_1(\alpha, \eta)$ returns to the initial value after η has circulated twice round the point $\eta = 0$ along a closed line C_2 as in Fig. 6a. In the first circulation η goes from the point 1 to 6, and in the second circulation from 6 to 1 (11).

The line C_2 encloses also the point $\eta = +1$ (a regular point of $K_1(\alpha, \eta)$), because the neighbourhood of $\eta = 0$ is cut from $\eta = -1$ to $\eta = +1$ by the transformation $\eta \rightleftarrows \alpha_d$ (4.2). In fact, the line $\mathscr{Q}_2$ is put on the two-sheeted Riemann surface η; the continuous line on the sheet η_I, the dashed on the sheet η_{II}. In the η-plane these two parts of the line cover each other.

Using the mapping described in Fig. 3 above, we find the trajectory of the zero-point α_d on the Riemann surface α_d for η moving along C_2. This is the line $\bar{C}_2$ in Fig. 6b. It begins at the point 1 on the sheet α_{II} and it returns to the initial point 1 (11), Fig. 6b. The dashed line indicates the part of the trajectory on the sheet α_{III}, the continuous line on the sheet α_{II}. The line $\bar{C}_2$ is also mapped into the β_d-plane – the line C_2 in Fig. 6c.

Having determined the trajectory α_d we can find the analytic continuation of the function $K_1(\alpha, \eta)$ (defined by (5.2b)) in the neighbourhood of $\eta = 0$. Three contours selected from eleven ones $Q = Q_i$, $i = 1, 2, \ldots, 11$, that result from pushing Q by the moving point α_d are shown in Fig. 7. The contour Q_i corresponds to the location α_d in the point i, $i = 1, 2, \ldots, 11$, as in Fig. 6b. This is the realization of analytic continuation of the Cauchy-type integral on Q. The $\alpha_d^{(1)}$, $\alpha_d^{(2)}$ points are additional, dependent on η branch points of the logarithm in this integral.

Looking at the trajectory of α_d, Fig. 6b, we can draw the contour Q_{11}. Although it is in the form of a very long loop it can be shown that the Cauchy-type integral on Q_{11} is equal to the Cauchy-type integral on $\mathscr{L}_2$ since the integral along a part of the contour "pushed" by $\alpha_d^{(1)}$ in first circulation is reduced by the integral along the same

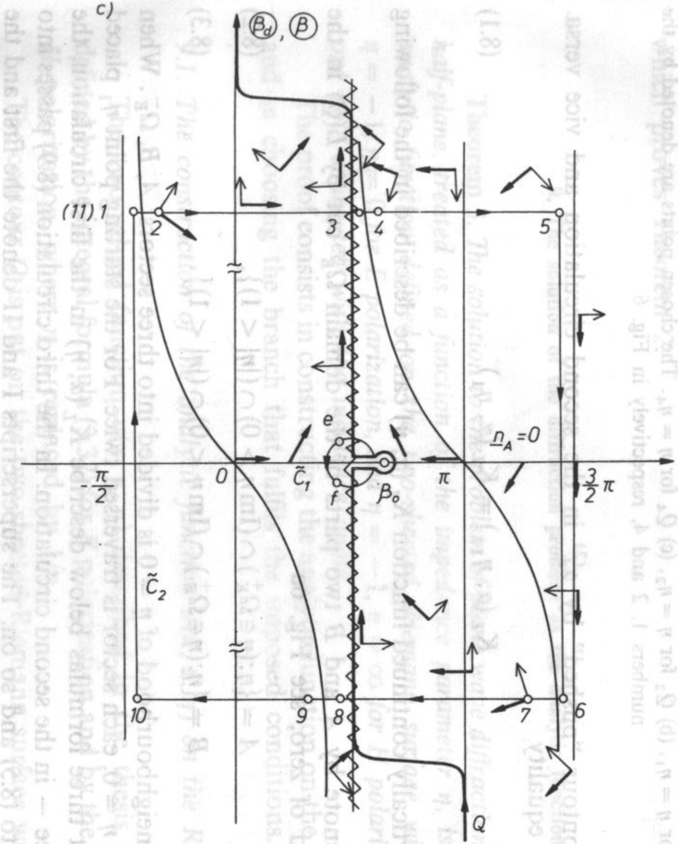


FIG. 6. (a) Two-sheeted Riemann surface η cut along the interval $[-1, +1]$ of the real axis. The line C_1 as the trajectory of η running arround $\eta = -1$ and the line C_2 as the trajectory arround $\eta = 0$ with $\eta = +1$ inside. Also the η-plane cut along $L(Q)$ with the sectors A, B, and Ω_E^- in the neighbourhood of $\eta = 0$ as described in Section 8; (b) The lines $\bar{C}_1$ and $\bar{C}_2$ as the mapping of C_1 and C_2 respectively into the Riemann surface α_d; (c) The lines $\tilde{C}_1$ and $\tilde{C}_2$ as the mapping of C_1 and C_2 respectively into β_d-plane. The vectors $\mathbf{n}_A \rightarrow$ and $\mathbf{n}_p \rightarrow$ characterise a surface wave described in Section 9. The outgoing wave is shown by $\mathbf{n}_P$ in the region $-\frac{\pi}{2} < \mathrm{Re}\,\beta \leq \frac{\pi}{2}$. The location of the contour Q corresponds to that in Fig. 4(b)

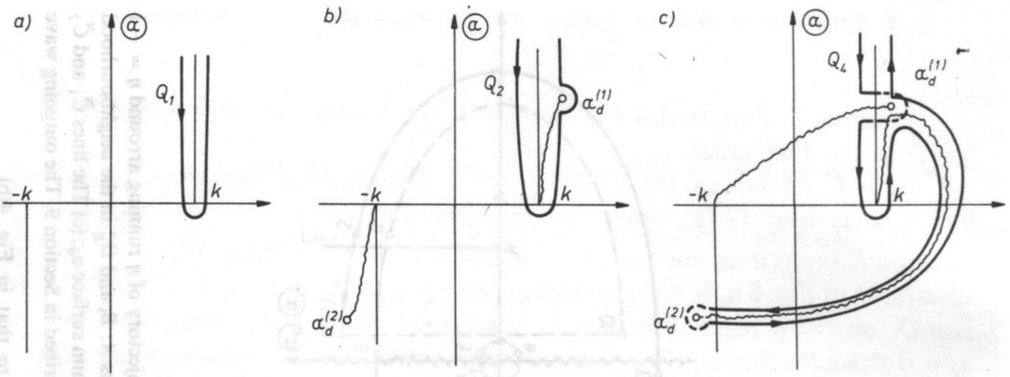


FIG. 7. The deformed contour Q as a realization of analytic continuation of $K_L(\alpha, \eta)$ in the neighbourhood of $\eta = 0$, (a) Q_1 for $\eta = \eta_1$, (b) Q_2 for $\eta = \eta_2$, (c) Q_4 for $\eta = \eta_4$. The chosen points are denoted by the numbers 1, 2 and 4, respectively in Fig. 6

part of the contour "pushed" by $\alpha_d^{(2)}$ in the second circulation and vice versa. Therefore the equality

$$K_L(\alpha, \eta_{11}) = K_L(\alpha, \eta_1) \tag{8.1}$$

holds.

The analytically continued function $K_L(\alpha, \eta)$ can be described in the following way:

Let us denote by A and B two parts of the domain Ω_E^+ cut by $L(Q)$ in the neighbourhood of zero, see Fig. 6a

$$A = \{\eta : (\eta \in \Omega_E^+) \cap (\operatorname{Im} \eta > 0) \cap (|\eta| < 1)\}, \tag{8.2}$$

$$B = \{\eta : (\eta \in \Omega_E^+) \cap (\operatorname{Im} \eta < 0) \cap (|\eta| < 1)\}. \tag{8.3}$$

Then the neighbourhood of $\eta = 0$ is divided into three sectors: A, B, Ω_E^-. When η goes around $\eta = 0$, each sector is traversed twice. For the starting point η_1 placed in Ω_E^- the first three formulas below describe $K_L(\alpha, \eta)$ in the first circulation, the remaining three – in the second circulation. In the third circulation (8.9) passes into (8.4), (8.4) – into (8.5) and so on. The superscripts I and II denote the first and the second branch of two-valued function $K_L(\alpha, \eta)$, respectively. We have for $\alpha \notin \Gamma_0$

$$K_L^{\mathrm{I}}(\alpha, \eta) = \exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta)}{t-\alpha}\, dt\right\} \text{for } \eta \in \Omega_E^-, \tag{8.4}$$

$$K_L^{\mathrm{I}}(\alpha, \eta) = \frac{\alpha - \alpha_d^{(1)}}{\alpha - k} \exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta) - 2\pi i}{t-\alpha}\, dt\right\} \text{for } \eta \in A, \tag{8.5}$$

$$K_L^{II}(\alpha, \eta) = \frac{\alpha - \alpha_d^{(2)}}{\alpha - k} \exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta) - 2\pi i}{t-\alpha}\, dt\right\} \text{for } \eta \in B, \tag{8.6}$$

$$K_L^{\mathrm{II}}(\alpha, \eta) = \frac{\alpha - \alpha_d^{(1)})(\alpha - \alpha_d^{(2)})}{(\alpha - k)^2} \times \tag{8.7}$$

$$\times \exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\ln K^*(t,\eta)-4\pi i}{t-\alpha}dt\right\} \text{for } \eta\in\Omega_E^-, \tag{8.7)[cd.]$$

$$K_L^{\mathrm{II}}(\alpha,\eta)=\frac{\alpha-\alpha_d^{(2)}}{\alpha-k}\exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\ln K^*(t,\eta)-2\pi i}{t-\alpha}dt\right\} \text{for } \eta\in A, \tag{8.8}$$

$$K_L^{\mathrm{I}}(\alpha,\eta)=\frac{\alpha-\alpha_d^{(1)}}{\alpha-k}\exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\ln K^*(t,\eta)-2\pi i}{t-\alpha}dt\right\} \text{for } \eta\in B, \tag{8.9}$$

In the formulas above the line Γ_0 is directed as in Fig. 5, $K^*(\alpha, \eta)$ is given by (7.2), and $\alpha_d^{(1)}$ and $\alpha_d^{(2)}$ are described by (4.3)–(4.5).

The derivation of the formulas (8.4)–(8.9) can be found in Appendix B.

This procedure of analytic continuation gives for $K_L(\alpha, \eta)$ exactly two branches which proves **Theorem** 2.

9. The solution of the diffraction problem as the analytic function of η

Theorem 3. *The solution to the problem of plane wave diffraction by an impedance half-plane, treated as a function of the impedance parameter η, is a branch of the analytic function defined on the two-sheeted Riemann surface with branch points $\eta = -1$, $\eta = 0$ for E polarization, and $\eta = -1$, $\eta = \infty$ for H polarization, except for one pole connected with the incidence angle.*

The proof consists in constructing the analytic function on the Riemann surface, and in choosing the branch that fulfils the imposed conditions.

9.1. *The construction of the analytic function $u(x, z; \eta)$ on the Riemann surface η*

The two-valued function $u(x, z; \eta)$ described in the thesis of **Theorem** 3 is defined by the integral formula (3.32) for E polarization and by the same formula for H polarization, where η is replaced by $1/\eta$.

The contour Q is put on the two-sheeted Riemann surface with branch points $\alpha = \pm k$, where the sheets are chosen as described in Section 3. The condition (3.7) requires that the branch cut pass through infinity, but its location Γ_0, Γ_π, as chosen here is directed only by the convenience of writing formulas for the amplitudes $A(\alpha, \eta)$, $B(\alpha, \eta)$ which makes the asymptotic analysis of the integral easier.

We impose two conditions: $\alpha_0 \in \Omega_Q^+$, and (3.7) for a location of Q. The condition (3.7) places the ends of Q on the real axis of the sheet α_{I}.

By fixing the contour Q, we get the first branch u_{I} of the function (3.32), where the integrand $F^{\mathrm{I}}(\alpha, \eta)$ is described by (3.33), (8.4), (8.5), (8.9) in the domain $|\eta| < 1$ for E polarization, and in the domain $|\eta| > 1$ for H polarization (if η is replaced by $1/\eta$ in the formula (3.32)).

In the second branch to be denoted by u^{II}, the contour of integration is a function of η. It is "pushed" by the pole $\alpha_d^{(2)}$ of the integrand. It can be reduced, however, to Q by "picking up" the pole at $\alpha = \alpha_d^{(2)}$.

Thus

$$u^{\mathrm{II}}(x, z; \eta) = u_i(x, z) - k\left\{\frac{1}{2\pi i}\int_Q F^{\mathrm{II}}(\alpha, \eta)e^{i\alpha x}e^{i\gamma|z|}d\alpha + \right.$$
$$\left. + \operatorname*{res}_{\alpha_d^{(2)}}[F^{\mathrm{II}}(\alpha, \eta)]e^{i\alpha_d^{(2)}x}e^{i\gamma_d^{(2)}|z|}\right\}, \qquad (9.1)$$

where $F^{\mathrm{II}}(\alpha, \eta)$ is described by (3.33), (8.6)–(8.8) in the domain $|\eta| < 1$ for E polarization, and in the domain $|\eta| > 1$ for H polarization when $1/\eta$ is substitutied for η.

This two-valued function on the η-plane, is single-valued on the two-sheeted Riemann surface, as shown in Fig. 8. We pass from one sheet to the other through the branch cut $L(Q)$.

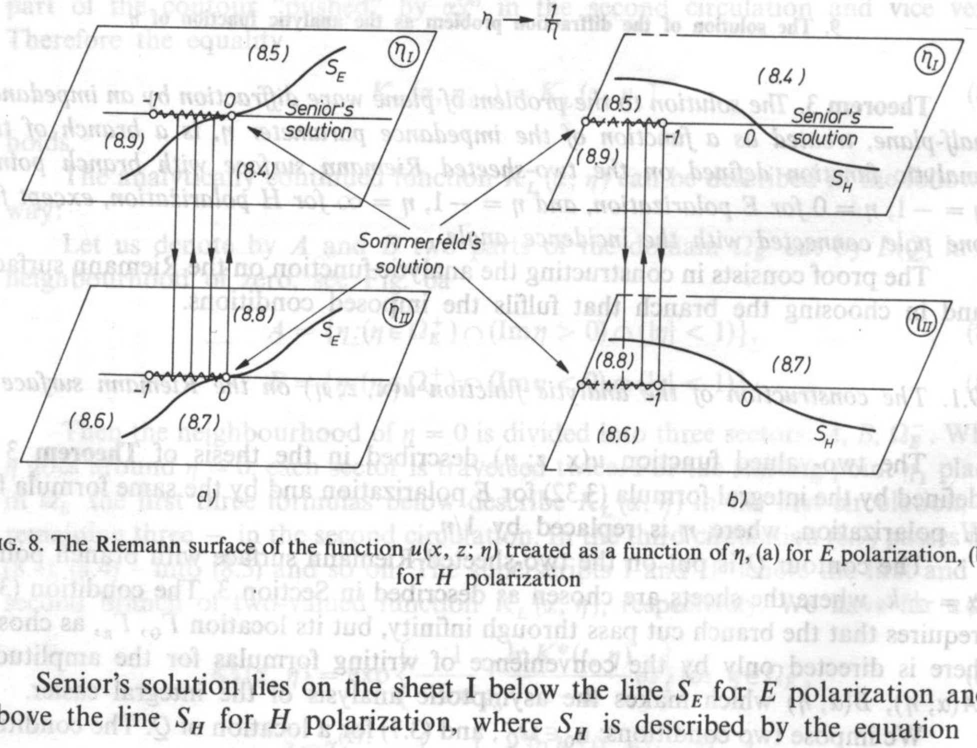


FIG. 8. The Riemann surface of the function $u(x, z; \eta)$ treated as a function of η, (a) for E polarization, (b) for H polarization

Senior's solution lies on the sheet I below the line S_E for E polarization and above the line S_H for H polarization, where S_H is described by the equation

$$\operatorname{Im}\eta = -\frac{\operatorname{Re}\eta}{\sqrt{1-(\operatorname{Re}\eta)^2}}. \qquad (9.2)$$

The lines S_E and S_H map one onto the other by the function $w = 1/\eta$.

The location of the contour Q corresponding to $L(Q)$ lying on the real axis is presented in Fig. 4b.

9.2. The properties of $u(x, z; \eta)$ as an analytic function of η

LEMMA. The point

$$\eta_0 = -\frac{1}{\sin\phi_0} \tag{9.3}$$

is a pole of the function $u(x, z; \eta)$ for E polarization, and the point

$$\eta_0 = -\sin\phi_0 \tag{9.4}$$

is a pole of the function $u(x, z; \eta)$ for H polarization.

Proof. We show the thesis for E polarization. Let us examine the function $F(\alpha, \eta)$ as given by (3.33) and extract the quotient $\dfrac{K_L(\alpha_0, \eta)}{K(\alpha_0, \eta)}$ in each branch of the function $u(x, z; \eta)$ in the sector A.

From (8.5) and (6.1a) we obtain for branch I

$$\frac{K_L^{\mathrm{I}}(\alpha_0, \eta)}{K(\alpha_0, \eta)} = \frac{\alpha_0 + k}{\alpha_0 - \alpha_d^{(2)}} \bar{K}(\alpha_0, \eta) \exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta) - 2\pi i}{t - \alpha_0}\, dt\right\}. \tag{9.5}$$

From (8.8) and (6.1a) we obtain for branch II

$$\frac{K_L^{\mathrm{II}}(\alpha_0, \eta)}{K(\alpha_0, \eta)} = \frac{\alpha_0 + k}{\alpha_0 - \alpha_d^{(1)}} \bar{K}(\alpha_0, \eta) \exp\left\{-\frac{1}{2\pi i}\int_{\Gamma_0} \frac{\ln K^*(t, \eta) - 2\pi i}{t - \alpha_0}\, dt\right\}. \tag{9.6}$$

The branch $u^{\mathrm{I}}(x, z; \eta) = u^{\mathrm{I}}(x, z; \alpha_d^{(2)})$ as described by (3.32), (3.33) and (9.5), and treated as a function of the parameter α_d, has the pole

$$\alpha_d^{(2)} = \alpha_0. \tag{9.7}$$

The branch $u^{\mathrm{II}}(x, z; \eta) = u^{\mathrm{II}}(x, z; \alpha_d^{(1)})$ as described by (3.32), (3.33) and (9.6), and treated as a function of the parameter α_d, has the pole

$$\alpha_d^{(1)} = \alpha_0 \tag{9.8}$$

Let us come back to the parameter η. From (9.7) the equality (9.3) for $0 \leqslant \phi_0 \leqslant \dfrac{\pi}{2}$ results, and vice versa. From (9.8) the equality (9.3) for $\dfrac{\pi}{2} \leqslant \phi_0 \leqslant \pi$ results and vice versa (this can be seen in Fig. 3 abg).

Since $u(x, z; \eta) \to \infty$ as $\eta \to \eta_0$ on the first or on the second sheet and since it is an analytic function in a ring neighbourhood of η_0, the point $\eta = \eta_0$ is a pole.

Summary: 1. The function $u(x, z; \eta)$ treated as a function of the impedance parameter has two algebraic branch points of the first order and one pole. These singular points are $\eta = 1$, $\eta = 0$, $\eta = \eta_0 = -1/[\sin\phi_0]$ for E polarization, and $\eta = -1$, $\eta = \infty$, $\eta = \eta_0 = -\sin\phi_0$ for H polarization.

2. Sommerfeld's solution is the limit of $u(x, z; \eta)$ at the branch point $\eta = 0$ (the limit (3.35) exists as η tends to zero in every section of the neighbourhood of $\eta = 0$).

3. If $\phi_0 \neq \frac{\pi}{2}$ then there exists a finite limit of $u(x, z; \eta)$ at the branch point $\eta = -1$. For $\phi_0 = \frac{\pi}{2}$ the limit is infinite.

9.3 *The properties of $u(x, z; \eta)$ as a function of x and z*

1) The edge condition: Each branch of $u(x, z; \eta)$ treated as a function of x and z belongs to the same class of functions, which derivatives (2.6) have the Fourier transforms tending to zero at infinity. From the half-range Fourier transforms (3.28), (3.29) we find

$$u_s(x, 0_+) - u_s(x, 0_-) = O(x^{1/2}) \quad \text{as } x \to 0_+, \tag{9.9}$$

and

$$\lim_{x \to 0_+} \left\{ \frac{\partial u_s}{\partial z}(x, 0_+) - \frac{\partial u_s}{\partial z}(x, 0_-) \right\} = -\frac{2ik}{\eta} \frac{K_L(\alpha_0, \eta)}{K(\alpha_0, \eta)}. \tag{9.10}$$

From the half-range Fourier transforms (3.30), (3.31) we find

$$\lim_{x \to 0} u_s(x, 0) = \frac{1}{K_U(\alpha_0, \eta)} \tag{9.11}$$

if the limit exists, and

$$\frac{\partial u}{\partial z}(x, 0) = O(x^{-1/2}) \tag{9.12}$$

as $x \to 0_-$.

2) The condition at infinity: In order to verify the condition at infinity, we change the contour Q in the formula (3.32) to the steepest descent path $\bar{S}(\theta)$, Fig. 9, where $0 < \theta < \pi$.

According to the Cauchy theorem and by applying Jordan's Lemma we have in the domain $0 < \theta < \pi$:

$$u_s^{\mathrm{I}}(x, z; \eta) = -k \left[\frac{1}{2\pi i} \Psi^{\mathrm{I}}(x, z; \eta) + H(\pi - \phi_0 - \theta) \operatorname*{res}_{\alpha_0} F^{\mathrm{I}}(\alpha, \eta)\, e^{i(\mathbf{n}_0 \cdot \boldsymbol{\varrho})} \right] \tag{9.13}$$

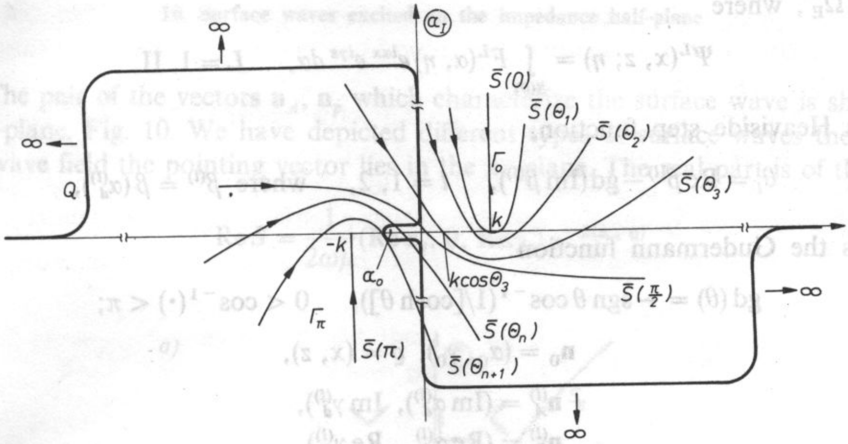


FIG. 9. The steepest descent paths. $\bar{S}(0)$ for $0 \leqq 0 \leqq \pi$ in α-plane cut along $\bar{S}(0)$, $\bar{S}(\pi)$ paths

for $\eta \in \Omega_E^-$, and

$$u_s^{\mathrm{I}}(x, z; \eta) = -k\Bigg[\frac{1}{2\pi i}\Psi^{\mathrm{I}}(x, z; \eta) + H(\pi-\phi_0-\theta)\operatorname*{res}_{\alpha_0} F^{\mathrm{I}}(\alpha, \eta)\, e^{i(\mathbf{n}_0\cdot\boldsymbol{\varrho})} + H(\theta_1-\theta)\operatorname*{res}_{\alpha_d^{(1)}} F^{\mathrm{I}}(\alpha, \eta)\, e^{-(\mathbf{n}_A^{(1)}\cdot\boldsymbol{\varrho})}\, e^{i(\mathbf{n}_P^{(1)}\cdot\boldsymbol{\varrho})}\Bigg] \tag{9.14}$$

for $\eta \in \Omega_E^+$.

$$u_s^{\mathrm{II}}(x, z; \eta) = -k\Bigg[\frac{1}{2\pi i}\Psi^{\mathrm{II}}(x, z; \eta) + H(\pi-\phi_0-\theta)\operatorname*{res}_{\alpha_0} F^{\mathrm{II}}(\alpha, \eta)\, e^{i(\mathbf{n}_0\cdot\boldsymbol{\varrho})} + H(\theta_2-\theta)\operatorname*{res}_{\alpha_d^{(2)}} F^{\mathrm{II}}(\alpha, \eta)\, e^{-(\mathbf{n}_A^{(2)}\cdot\boldsymbol{\varrho})}\, e^{i(\mathbf{n}_P^{(2)}\cdot\boldsymbol{\varrho})}\Bigg] \tag{9.15}$$

for $\eta \in \Omega_E^+$, and

$$u_s^{\mathrm{II}}(x, z; \eta) = -k\Bigg[\frac{1}{2\pi i}\Psi^{\mathrm{II}}(x, z; \eta) + H(\pi-\phi_0-\theta)\operatorname*{res}_{\alpha_0} F^{\mathrm{II}}(\alpha, \eta)\, e^{i(\mathbf{n}_0\cdot\boldsymbol{\varrho})} + \operatorname*{res}_{\alpha_d^{(2)}} F^{\mathrm{II}}(\alpha, \eta)\, e^{-(\mathbf{n}_A^{(2)}\cdot\boldsymbol{\varrho})}\, e^{i(\mathbf{n}_P^{(2)}\cdot\boldsymbol{\varrho})}\Bigg] \tag{9.16}$$

for $\eta \in \Omega_E^-$, where

$$\Psi^L(x, z; \eta) = \int_{S(\theta)} F^L(\alpha, \eta)\, e^{i\alpha x}\, e^{i\gamma z}\, d\alpha, \quad L = \mathrm{I, II}, \tag{9.17}$$

$H(\cdot)$ is Heaviside step function,

$$\theta_l = \operatorname{Re}\beta^{(l)} - \operatorname{gd}(\operatorname{Im}\beta^{(l)}), \quad l = 1, 2, \quad \text{where } \beta^{(l)} = \beta(\alpha_d^{(l)}),$$

$\operatorname{gd}(\theta)$ is the Gudermann function:

$$\operatorname{gd}(\theta) = -\operatorname{sgn}\theta \cos^{-1}(1/[\cosh\theta]), \quad 0 < \cos^{-1}(\cdot) < \pi;$$

$$\mathbf{n}_0 = (\alpha_0, \gamma_0), \; \varrho = (x, z),$$

$$\mathbf{n}_A^{(l)} = (\operatorname{Im}\alpha_d^{(l)}), \; \operatorname{Im}\gamma_d^{(l)}),$$

$$\mathbf{n}_P^{(l)} = (\operatorname{Re}\alpha_d^{(l)}, \; \operatorname{Re}\gamma_d^{(l)})$$

$$\text{for } l = 1, 2. \tag{9.18}$$

On evaluating (9.17) for large $k\varrho$ we obtain

$$\int_{S(\theta)} F^L(\alpha, \eta)\, e^{i\alpha x}\, e^{i\gamma z}\, d\alpha \sim \sqrt{\frac{2\pi}{k\varrho}} F^L(k\cos\theta)\, e^{i(k\varrho - \pi/4)} \quad \text{for } L = \mathrm{I, II}. \tag{9.19}$$

Each function represents the cylindrical wave which decays exponentially at infinity if small positive imaginary part is inserted in k.

The residue term at α_0 gives rise to the reflected wave. The residue term at $\alpha_d^{(l)}$, $l = 1, 2$ gives rise to the nonhomogeneous plane wave called here a surface wave. The surface wave propagates in the direction defined by $\mathbf{n}_p$, and its amplitude decays in the direction defined by $\mathbf{n}_A$.

For some selected points of the η–Riemann surface, as in Fig. 6a, the corresponding pairs of the vectors $\mathbf{n}_A$, $\mathbf{n}_p$ are shown in the β–plane, Fig. 6c. The components of the vectors are found using (9.18) and the mappings in Fig. 3.

A surface wave generated by the pole lying in the strip $-\frac{\pi}{2} < \operatorname{Re}\beta \leqq \frac{\pi}{2}$ belongs to u_s^{I}. It is an outgoing wave. It propagates in an angular domain from the edge into the half-space $x > 0$. A surface wave generated by the pole lying in the strip $\frac{\pi}{2} < \operatorname{Re}\beta \leqq \frac{3}{2}\pi$ belongs to u_s^{II} and can be treated as an incoming one. As could be expected, the outgoing wave condition is fulfiled for u_s^{I} only.

In conclusion we state that the branch u^{I} of the function as shown in Fig. 8, is the solution to our diffraction problem for every η. This completes the proof of Theorem 3.

10. Surface waves excited on the impedance half-plane

The pair of the vectors $\mathbf{n}_A$, $\mathbf{n}_p$ which characterize the surface wave is shown in the η–plane, Fig. 10. We have depicted different types of surface waves there. For each wave field the pointing vector lies in the xz plane. The real part is of the form

$$\operatorname{Re} S = \frac{1}{2\omega\mu}(\operatorname{Re}\alpha_d,\, 0,\, \operatorname{Re}\gamma_d)\, e^{-2(\mathbf{n}_A\cdot\mathbf{\varrho})} \tag{10.1}$$

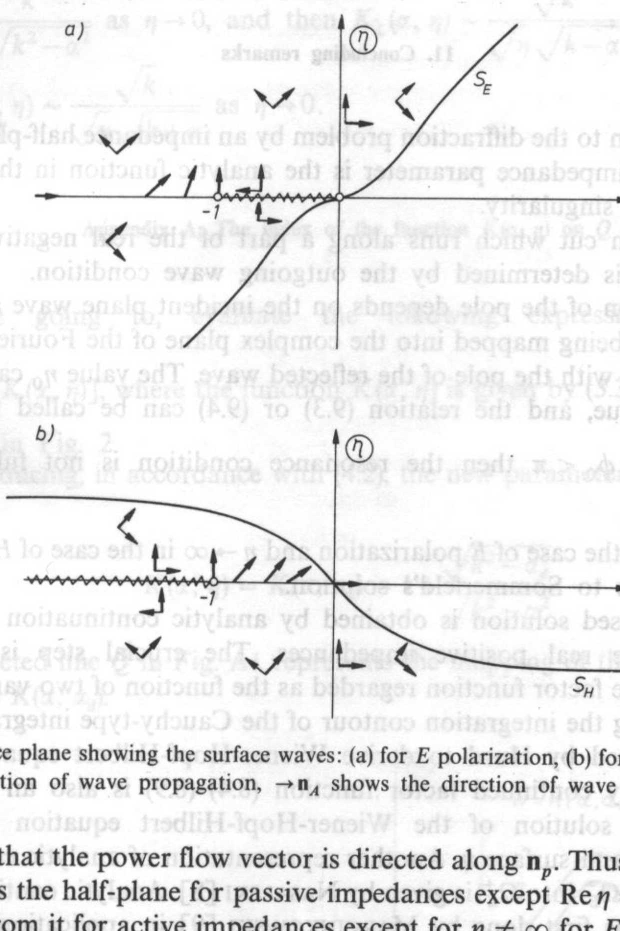


FIG. 10. Impedance plane showing the surface waves: (a) for E polarization, (b) for H polarization, $\rightarrow \mathbf{n}_p$ shows the direction of wave propagation, $\rightarrow \mathbf{n}_A$ shows the direction of wave amplitude decaying

which implies that the power flow vector is directed along $\mathbf{n}_p$. Thus, the wave carries energy towards the half-plane for passive impedances except $\operatorname{Re}\eta = 0$, and it carries energy away from it for active impedances except for $\eta \neq \infty$ for E–polarization and $\eta \neq 0$ for H–polarization. In these exceptional cases, the power-vector is parallel to the half-plane.

The direction of $\mathbf{n}_A$–vector shows, that the amplitude of the surface wave exponentially decays in the directions of both x and z–axis for passive impedances, $\operatorname{Re}\eta = 0$ excluded, while for active impedances the situation is different. The amplitude exponentially grows in one direction and decays in the other direction. One vector component is negative, the second one is positive, and they change the signs on the real negative semiaxis.

One more difference between passive and active impedances is, that no surface wave exists for passive impedances belonging to the domain bounded by S_E and S_H lines (see also BOWMAN [6]). On the contrary, for active impedances belonging to the domain bounded by S_E and S_H lines there exists a surface wave of E polarization as well as that of H polarization.

11. Concluding remarks

1. The solution to the diffraction problem by an impedance half-plane treated as a function of the impedance parameter is the analytic function in the cut η–plane except for a pole singularity.

2. The branch cut which runs along a part of the real negative axis in the complex η–plane is determined by the outgoing wave condition.

3. The location of the pole depends on the incident plane wave angle ϕ_0. The pole $\eta = \eta_0$, after being mapped into the complex plane of the Fourier transformed solution, coincides with the pole of the reflected wave. The value η_0 can be regarded as a resonant value, and the relation (9.3) or (9.4) can be called the resonance condition. If $\frac{\pi}{2} < \phi_0 < \pi$ then the resonance condition is not fulfiled for any impedance.

4. As $\eta \to 0$ in the case of E polarization and $\eta \to \infty$ in the case of H polarization, the solution tends to Sommerfeld's solution.

5. The discussed solution is obtained by analytic continuation of the Senior solution from the real positive impedances. The crucial step is the analytic continuation of the factor function regarded as the function of two variables. This is done by deforming the integration contour of the Cauchy-type integral. Earlier, the same idea was used by Hurd to derive Wiener-Hopf-Hilbert equations.

6. Analytically continued factor function (8.4)–(8.9) is also an analytic continuation of the solution of the Wiener-Hopf-Hilbert equation on Γ_0, onto two-sheeted Riemann surface η. Another representation of analytic continuation of this solution from Ω_E^- into Ω_E^+ is given by NASALSKI [8]. Analytic continuation of the factor function was first done by MARCINKOWSKI [9] in application to numerical calculations.

7. The function $K_L(\alpha, \eta)$ can be expressed in terms of Maliuzhinets function [10]. Such representation is suitable for computations which are performed in different ways by several authors. The most complete information can be found in [11].

8. It should be noted, that the function $K(\alpha, \eta)$ used in this paper differs by the

factor $1/\eta$ from the function $K(\alpha)$ introduced by SENIOR [1], that is $K(\alpha, \eta) = \frac{1}{\eta} K(\alpha)$, hence $K_L(\alpha, \eta) = \frac{1}{\sqrt{\eta}} K_-(\alpha)$, $K_U(\alpha, \eta) = \frac{1}{\sqrt{\eta}} \cdot \frac{1}{K_+(\alpha)}$.

9. It should be also emphasized, that while the solution to the diffraction problem for the impedance half-plane tends to Sommerfeld's solution as $\eta \to 0$, the factorized function used here does not tend to the factorized function in Sommerfeld's problem. The same refers to the factor functions. That is:

$$K(\alpha, \eta) \sim \frac{k}{\eta\sqrt{k^2-\alpha^2}} \text{ as } \eta \to 0, \text{ and then } K_L(\alpha, \eta) \sim \frac{\sqrt{k}}{\sqrt{\eta}\sqrt{k-\alpha}},$$

$$K_U(\alpha, \eta) \sim \frac{\sqrt{k}}{\sqrt{\eta}\sqrt{k+\alpha}} \text{ as } \eta \to 0.$$

Appendix A. The index of the function $K(\alpha, \eta)$ on Q

We are going to, evaluate the following expression: $\operatorname{ind}_Q K(\alpha, \eta) = \frac{1}{2\pi}\int_Q d[\arg K(\alpha, \eta)]$, where the function $K(\alpha, \eta)$ is given by (3.32) and the contour Q is shown in Fig. 2.

By introducing, in accordance with (4.2), the new parameter α_d instead of η we have

$$K(\alpha, \eta) = K(\alpha, \alpha_d) = 1 + \frac{\sqrt{k^2-\alpha_d^2}}{\sqrt{k^2-\alpha^2}} \tag{A1}$$

The directed line $\bar{Q}$ in Fig. A1 represents the mapping of the contour Q by the function $w = K(\alpha, \alpha_d)$.

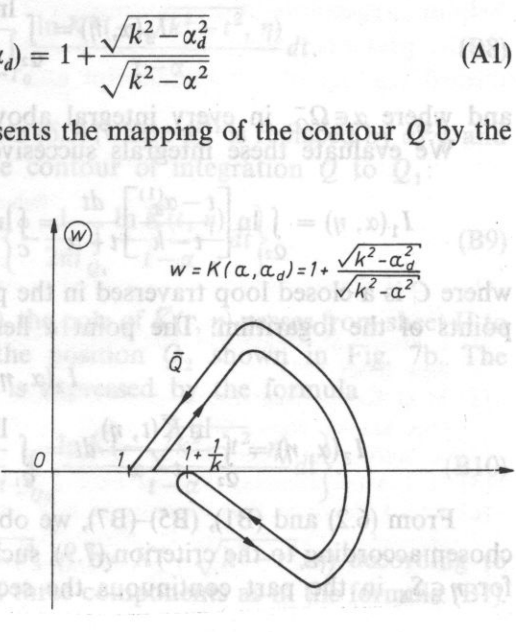


FIG. A1. The directed line $\bar{Q}$ as the mapping of the contour Q by the function $K(\alpha, \eta) = K(\alpha, \alpha_d)$

The increase of the argument of $K(\alpha, \alpha_d)$ on the contour Q is equal to the increase of the argument of the vector $w(Q)$, the initial point of which is $(0, 0)$, and the end-point moves along the directed line $\bar{Q}$. This increase can be easily read from the picture, and is equal to zero. In conclusion $\operatorname{ind}_Q K(\alpha, \eta) = 0$

Appendix B. Derivation of the formulas (8.4) – (8.9)

The first two formulas (8.4) and (8.5), are already known as coinciding with (7.1ab). We shall derive (8.5) once again by using a different method which also works in each of the remaining sectors of η.

For $K_L(\alpha, \eta)$, we use the representation (6.2), where the contour Q shifted to the location Q_2 is shown in Fig. 7b. We write the Cauchy-type integral as a sum of three components

$$\int_{Q_2} \ln\left[\frac{(t-\alpha_d^{(1)})(t-\alpha_d^{(2)})\,\bar{K}(t, \eta)}{(t+k)(t-k)}\right]\frac{dt}{t-\alpha} = I_1(\alpha, \eta) + I_2(\alpha, \eta) + I_3(\alpha, \eta), \tag{B1}$$

where

$$I_1(\alpha, \eta) = \int_{Q_2} \ln\left[\frac{t-\alpha_d^{(1)}}{t-k}\right]\frac{dt}{t-\alpha}, \tag{B2}$$

$$I_2(\alpha, \eta) = \int_{Q_2} \ln\left[\frac{t-\alpha_d^{(2)}}{t+k}\right]\frac{dt}{t-\alpha}, \tag{B3}$$

$$I_3(\alpha, \eta) = \int_{Q_2} \frac{\ln \bar{K}(t, \eta)}{t-\alpha}\,dt, \tag{B4}$$

and where $\alpha \in \Omega_{Q_2}^-$ in every integral above.

We evaluate these integrals succesively.

$$I_1(\alpha, \eta) = \int_{Q_2} \ln\left[\frac{t-\alpha_d^{(1)}}{t-k}\right]\frac{dt}{t-\alpha} = \oint_C \ln\left[\frac{t-\alpha_d^{(1)}}{t-k}\right]\frac{dt}{t-\alpha} = -2\pi i \ln\frac{\alpha-\alpha_d^{(1)}}{\alpha-k}, \tag{B5}$$

where C is a closed loop traversed in the positive direction, containing both branch points of the logarithm. The point α lies outside of C.

$$I_2(\alpha, \eta) = 0, \tag{B6}$$

$$I_3(\alpha, \eta) = \int_{Q_2} \frac{\ln \bar{K}(t, \eta)}{t-\alpha}\,dt = \int_{Q_1} \frac{\ln \bar{K}(t, \eta)}{t-\alpha}\,dt = \int_{\Gamma_0} \frac{\ln K^*(t, \eta)}{t-\alpha}\,dt. \tag{B7}$$

From (6.2) and (B1), (B5)–(B7), we obtain (8.5). The branch of the logarithm is chosen according to the criterion (7.9), such that the function $K_L(\alpha, \eta)$ is continuous for $\eta \in S_E$ in the part continuous the sector A, Fig. B1.

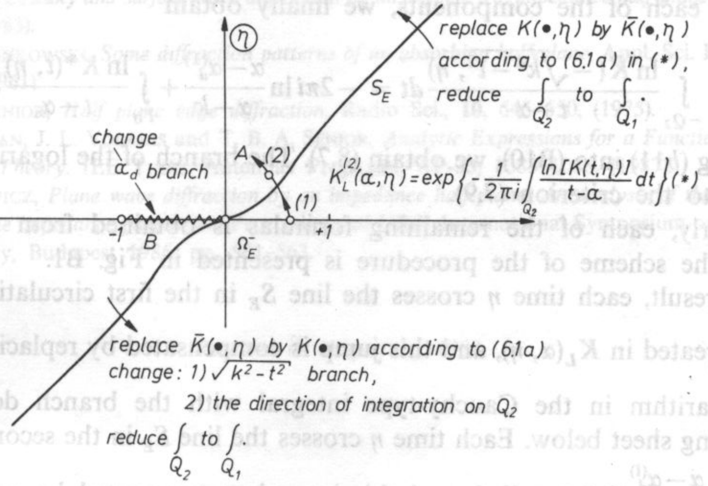


FIG. B1. The scheme of evaluation of $K_L(\alpha, \eta)$ in the neighbourhood of $\eta = 0$, while η goes around $\eta = 0$, starting from the point (1)

The formula (8.6) is obtained from (8.5) by changing the branch α_d thus leading to the continuity of the function $K_L(\alpha, \eta)$ for $\eta \in [-1, 0)$, Fig. B1.

For $\eta \in \Omega_E^-$ now, the contour Ω partially lies on the sheet α_{II}, Fig. 7c. We are going to continue analytically the function (8.6)

If we change $\sqrt{k^2-t^2}$ to $-\sqrt{k^2-t^2}$ in (8.6), we have to change the direction of integration on Γ_0 according to the equality

$$\int_{\Gamma_0} \frac{\ln K^*(\sqrt{k^2-t^2}, \eta)}{t-\alpha} dt = \int_{\Gamma_0} \frac{\ln K^*(-\sqrt{k^2-t^2}, \eta)}{t-\alpha} dt. \tag{B8}$$

We rewrite the formula (8.6) in the form (6.3) (where $\alpha_d^{(1)}$ is replaced by $\alpha_d^{(2)}$), and continue it analytically by deforming the contour of integration Q to Q_1:

$$K_L(\alpha, \eta) = \frac{\alpha - \alpha_d^{(2)}}{\alpha - k} \exp\left\{-\frac{1}{2\pi i}\int_{Q_1} \frac{\ln \bar{K}(t, \eta)}{t-\alpha} dt\right\} \tag{B9}$$

When η passes from B to Ω_E^- (Fig. B1), the pole of $\bar{K}(t, \eta)$ passes from sheet II to sheet I. It pushes the contour Q_1 to the position Q_2 shown in Fig. 7b. The analytically continued function $K_L(\alpha, \eta)$ is expressed by the formula

$$K_L(\alpha, \eta) = \frac{\alpha - \alpha_d^{(2)}}{\alpha - k} \exp\left\{-\frac{1}{2\pi i}\int_{-Q_2} \frac{\ln \bar{K}(-\sqrt{k^2-t^2}, \eta)}{t-\alpha} dt\right\}. \tag{B10}$$

We replace the function $\bar{K}(-\sqrt{k^2-t^2}, \eta)$ by $K(-\sqrt{k^2-t^2}, \eta)$ according to (6.1a), and rewrite the integral as a sum of three components as in the formula (B1).

Evaluating each of the components, we finally obtain

$$\int_{-Q_2} \frac{\ln \bar{K}(-\sqrt{k^2-t^2}, \eta)}{t-\alpha} dt = -2\pi i \ln \frac{\alpha-\alpha_d^{(1)}}{\alpha-k} + \int_{\Gamma_0} \frac{\ln K^*(t, \eta)}{t-\alpha} dt. \tag{B11}$$

Putting (b11) into (B10), we obtain (8.7). The branch of the logarithm is chosen according to the criterion (7.9).

Similarly, each of the remaining formulas is obtained from the preceding formula. The scheme of the procedure is presented in Fig. B1.

As a result, each time η crosses the line S_E in the first circulation, the factor $\dfrac{\alpha-\alpha_d^{(i)}}{\alpha-k}$ is created in $K_L(\alpha, \eta)$, and this jump is compensated by replacing the branch of the logarithm in the Cauchy-type integral with the branch defined on the neighbouring sheet below. Each time η crosses the line S_E in the second circulation, the factor $\dfrac{\alpha-\alpha_d^{(i)}}{\alpha-k}$ is cancelled, and this jump is compensated by replacing of the branch of logarithm with that defined on the neighbouring sheet above.

After the two circulations the value of the analytically continued function $K_L(\alpha, \eta)$ returns to the initial value.

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