PROPERTIES OF A SOUND SOURCE WITH GAUSS VELOCITY AMPLITUDES DISTRIBUTION. FAR FIELD

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1. Introduction

It is well known that the uniform velocity distribution on a baffled piston results in a directivity pattern with lateral lobes. To improve that directivity, it is necessary to look for variable velocity distribution. The first one computed was Gauss curve distribution [8]. The directivity index has the form of the Hankel transform (zero order) of the velocity distribution [6]. For that reason the Gauss distribution has no lateral lobes in the directivity pattern — the transform of Gauss curve is a Gauss curve, too. In the paper [7] we have computed approximately for that source the directivity pattern R, the real part of relative impedance θ and the imaginary part χ. We denote by "a" the radius of a baffled piston. The velocity amplitude distribution $u(r)$ (r — polar coordinate on the piston) takes then the form

$$u(r) = u_0 \cdot e^{-(\frac{n \cdot r}{a})^2}, \tag{1}$$

In the above formula "n" denotes a gauge factor. In the paper [7] the approximate method of computations used was based on the assumption that the contribution to the value of the integral transform for $r > a$ is negligible. In that situation we can extend the corresponding integrals to infinity. Of course the accuracy of the approximation is better when n is greater. In that manner we obtain integrals easy to be computed in a closed analytical form.

In the present paper we give the accurate computations — we compute the corresponding integrals in the limits from 0 to "a" and compare the results with the appoximate ones. For that reason we accept the following velocity distribution on the piston

$$u(r) = \begin{cases} u_0 \cdot e^{-(\frac{n \cdot r}{a})^2} & 0 \le r \le a \\ 0 & r > a \end{cases}. \tag{2}$$

Nevertheless, to maintain the continuity of our reasoning we begin by reminding the basic formulae of the approximate theory [7]. This is necessary also from another point of view. In the paper [7] we did not use the gauge factor n and the corresponding formulae must now we written in a corrected form. The near field of such a source was computed in [9].

2. Theoretical basis of computations and summary of the approximate method

According to the paper [6], the directivity pattern of a source with symmetry center and the velocity distribution $u(r)$ has the form

$$R(ka \cdot \sin \gamma) = \frac{2 \cdot \pi}{Q} \int_0^a u(r) \cdot J_0(kr \cdot \sin \gamma) \cdot r \cdot dr. \tag{3}$$

In the above formula Q denotes the volume output of the source. $J_0()$ is the Bessel function of zero order, k the wave number, γ the angle between the field point direction and the "z" axis perpendicular to the piston at its center. The formula (3) represents the Hankel transform of zero order of the function $u(r)$ [8] in the case when $u(r)$ equals 0 for $r > a$. In our case (2) the directivity pattern equals

$$R(ka \cdot \sin \gamma) = \frac{2 \cdot \pi \cdot u_0}{Q} \int_0^a e^{-(\frac{n \cdot r}{a})^2} J_0(kr \cdot \sin \gamma) \cdot r \cdot dr, \tag{4}$$

If we take the upper limit of the integral (4) "∞" instead of "a", we get the approximate value

$$R(ka \cdot \sin \gamma) = \frac{2 \cdot \pi \cdot u_0}{Q} \int_0^\infty e^{-(\frac{n \cdot r}{a})^2} \cdot J_0(kr \cdot \sin \gamma) \cdot r \cdot dr. \tag{5}$$

The source output Q equals (for the approximate case)

$$Q = 2 \cdot \pi \cdot u_0 \int_0^\infty e^{-(\frac{n \cdot r}{a})^2} \cdot r \cdot dr. \tag{6}$$

Computing the elementary integral (6), we get

$$Q = \pi \cdot u_0 \cdot \left(\frac{a}{n}\right)^2. \tag{7}$$

The integral in the formula (3) is given in tables of integrals [2 p. 731] and with Eq. (7) we get the directivity index (5) in the form

$$R(ka \cdot \sin \gamma) = e^{-(\frac{ka \cdot \sin \gamma}{2n})^2}. \tag{8}$$

It is, as we may expect, a Gauss distribution, too.

As we know, by increasing "n" we decrease the error of the approximate method, but simultaneously we deteriorate the directivity. We will see it exactly, later on, in the discussion of the results.

Now we begin the computation of the real part θ of the specific impedance of the above considered source and its imaginary part χ. In the case of a uniform velocity distribution we have [6]

$$\theta(ka) = \frac{k^2 \cdot S}{2 \cdot \pi} \cdot \int_0^{\frac{\pi}{2}} R^2(ka \cdot \sin \gamma) \cdot \sin \gamma \cdot d\gamma, \tag{9}$$

where S denotes the area of the piston. In the case of variable velocity distribution we

must introduce in the formula (9) a coefficient κ normalizing $\lim_{ka\to\infty} \theta$ to unity. Then

$$\theta(ka) = \frac{k^2 \cdot S}{2 \cdot \pi} \cdot \kappa \cdot \int_0^{\frac{\pi}{2}} R^2(ka \cdot \sin\gamma) \cdot \sin\gamma \cdot d\gamma. \tag{10}$$

It is sufficient to examine the classic reasoning leading to the formula (9) to understand that it was computed by equating the acoustic power radiated on the source to the entire power in the far field. In the first case, that power expressed by θ is proportional to the mean value of the square of velocity amplitude. In the second case it is proportional to the square of the output, i.e., to the square of the mean value of the velocity amplitude. Therefore the coefficient κ equals

$$\kappa = \frac{(u_m)^2}{(u)^2_m} \tag{11}$$

Sometimes it is possible to compute κ immediately from the condition

$$\lim_{ka\to\infty} \theta(ka) = 1. \tag{12}$$

Of course the formula (10) is a general one and, depending on the substitution — accurate or approximate value of $R(ka \sin\gamma)$, we get the accurate or approximate value of $\theta(ka)$.

We know two methods of computing the imaginary part from the real part θ. The first one consists in computing the Hilbert transform of the real part:

$$\chi(ka) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\theta(x)}{x - ka} \cdot dx. \tag{13}$$

The second one, developed by RDZANEK [5] consists in substituting, in the formula (10) $\cosh\psi$ instead of $\sin\gamma$ and integrating from 0 to ∞.

$$\chi(ka) = \frac{k^2 \cdot S}{2 \cdot \pi} \cdot \int_0^\infty R^2(ka \cdot \cosh\psi) \cdot \cosh\psi \, d\psi. \tag{14}$$

The formulae (13) and (14) give us an accurate or approximate value depending on the accepted value of R. Returning to the approximate method, we confine our computations to the results given in the paper [7]. We have the coefficient computed from the formula (11) in the form

$$\kappa = \frac{2}{n^2}. \tag{15}$$

The following formulae are simpler if we introduce the so-called diffraction parameter w

$$w = \frac{ka}{\sqrt{2}}. \tag{16}$$

We write, according to (7)

$$\theta(w) = 2 \cdot \frac{w}{n} \cdot D\left(\frac{w}{n}\right) \tag{17}$$

where the so-called Dawson integral or Dawson formula [1] is given in the form

$$D(x) = e^{-x^2} \cdot \int_0^x \cdot e^{t^2} \cdot dt \tag{18}$$

The imaginary part of the specific impedance was computed in [7] by means of the Hilbert transform (13) and has the form

$$\chi = \sqrt{\pi} \cdot \frac{w}{n} \cdot e^{-\left(\frac{w}{n}\right)^2} \qquad (19)$$

The calculated values of R, θ and χ (for the approximated method) will be shown in corresponding figures together with the accurate ones. An interesting result is obtained by equating Eq. (19) to (14); of course we must substitute in the last one the value (16) for κ. Writing x instead of ψ, we get

$$\int_0^\infty e^{-t^2 \cdot \cosh^2 x} \cdot \cosh x \cdot dx = \frac{\sqrt{\pi}}{t} \cdot e^{-t^2}. \qquad (20)$$

The formula (20) gives a new definite integral, until now not computed and not given in any tables.

3. Accurate method — directivity index

Our starting point is now the formula (4) where we have the source output Q. That output must be computed exactly i.e., instead of the formula (6) we write the analogous integral but in the limits from "0" to "a". We obtain in that way

$$Q = 2 \cdot \pi \cdot u_0 \int_0^a e^{-\left(\frac{n \cdot r}{a}\right)^2} \cdot r \cdot dr \qquad (21)$$

The integral in (21) is an elementary one and owing to that output Q takes the form

$$Q = \pi \cdot u_0 \cdot \left(\frac{a}{n}\right)^2 \cdot (1 - e^{-n^2}) \qquad (22)$$

Substituting Eq. (22) to the formula (4) for the directivity index, we get it in the form

$$R(ka \cdot \sin \gamma) = \frac{2 \cdot \left(\frac{n}{a}\right)^2}{1 - e^{-n^2}} \int_0^a e^{-\left(\frac{n \cdot r}{a}\right)^2} \cdot J_0(kr \cdot \sin \gamma) \cdot r \cdot dr \qquad (23)$$

In the formula (23) we introduce a new variable

$$x = \frac{r}{a}, \qquad (24)$$

and get

$$R(ka \cdot \sin \gamma) = \frac{2 \cdot n^2}{1 - e^{-n^2}} \int_0^1 e^{-n^2 \cdot x^2} \cdot J_0(ka \cdot \sin \gamma \cdot x) \cdot x \cdot dx \qquad (25)^*$$

The r.h.s. of the formula (25) may be integrated. To simplify the integration we denote

$$ka \cdot \sin \gamma = b. \qquad (26)$$

and

$$x = \frac{1}{b} \cdot y. \qquad (27)$$

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* The author is grateful to dr J. Janczar for analytical integration of (25).

Then:

$$R(b) = \frac{2 \cdot n^2}{(1 - e^{-n^2})b^2} \int_0^b e^{\frac{-n^2}{b^2} \cdot y^2} y \cdot J_0(y) \, dy. \tag{28}$$

We use the integration by parts and obtain:

$$R(b) = \frac{1}{e^{n^2} - 1} \sum_{m=0}^{\infty} (2n^2)^{m+1} \cdot \frac{J_{m+2}(b)}{b^{m+1}}. \tag{29}$$

One must check whether the directivity pattern equals "one" for $\sin \gamma = 0$. In that case we have $b = 0$ and [2]:

$$\lim_{b \to \infty} \frac{J_{m+2}(b)}{b^{m+1}} = \frac{1}{(n+1)! \cdot 2^{m+1}}. \tag{30}$$
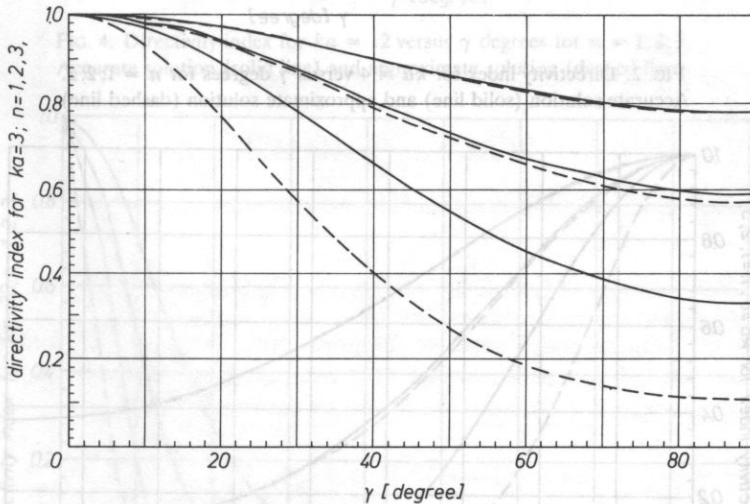
FIG. 1. Directivity index for $ka = 3$ versus γ degrees for $n = 1; 2; 3$. Accurate solution (solid line) and approximate solution — (dashed line).

The directivity pattern equals then:

$$R(0) = \frac{1}{e^{n^2} - 1} \sum_{m=1}^{\infty} \frac{(n^2)^m}{m!} = \frac{1}{e^{n^2} - 1}(e^{n^2} - 1) = 1. \tag{31}$$

Of course we have for $b \ll 1$

$$R(b) \approx 1. \tag{32}$$

For $b \gg 1$ and $b > 2n^2$ we have approximately:

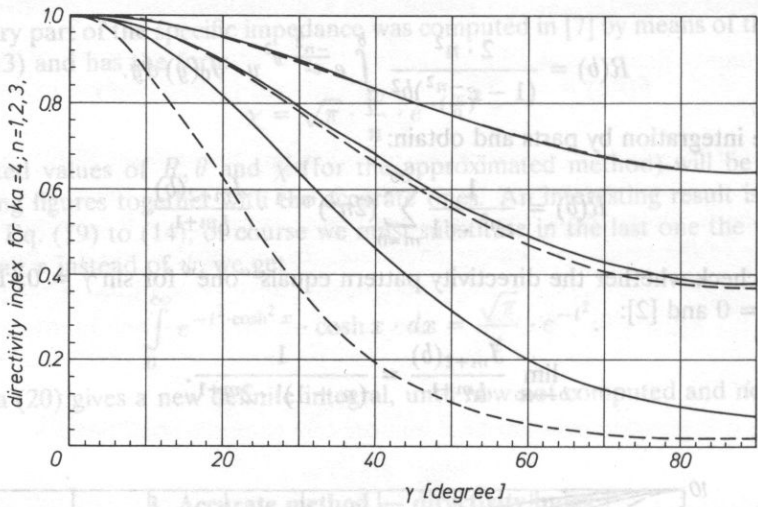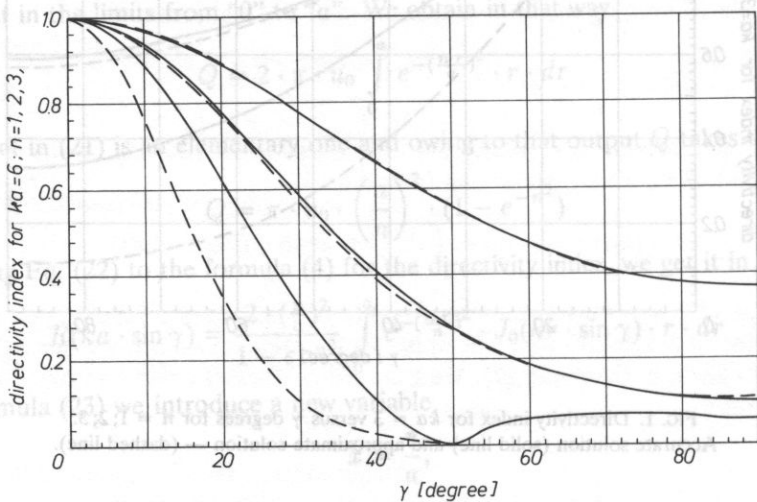$$R(b) \approx \frac{2n^2}{e^{n^2} - 1} \cdot \frac{J_1(b)}{b}, \tag{33}$$

FIG. 2. Directivity index for $ka = 4$ versus γ degrees for $n = 1; 2; 3$. Accurate solution (solid line) and approximate solution (dashed line).

FIG. 3. Directivity index for $ka = 6$ versus γ degrees for $n = 1; 2; 3$. Accurate solution (solid line) and approximate solution (dashed line).

or finally:

$$R(ka \sin \gamma) \approx \frac{2n^2}{e^{n^2} - 1} \cdot \frac{J_1(ka \sin \gamma)}{ka \sin \gamma}. \tag{34}$$

Of course, for convenience we may always write formally:

$$R(ka \sin \gamma) = \frac{R^\infty(ka \cdot \sin \gamma)}{1 - e^{-n^2}} \cdot F(n, ka \cdot \sin \gamma), \tag{35}$$
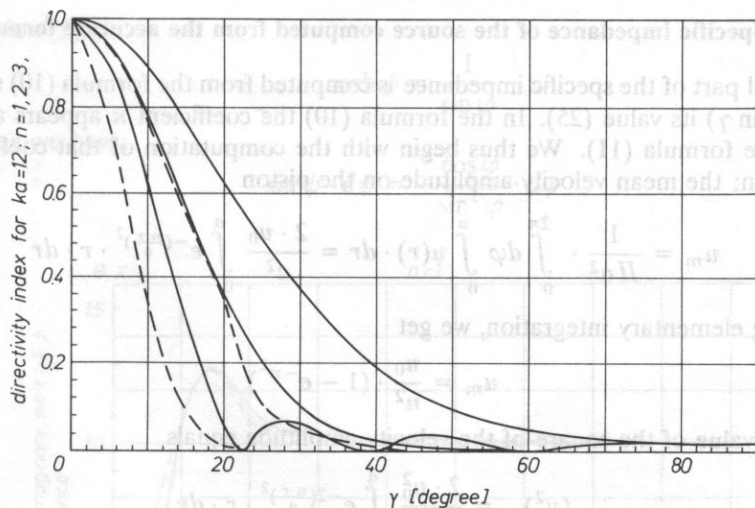
FIG. 4. Directivity index for $ka = 12$ versus γ degrees for $n = 1; 2; 3$.
Accurate solution (solid line) and approximate solution (dashed line).
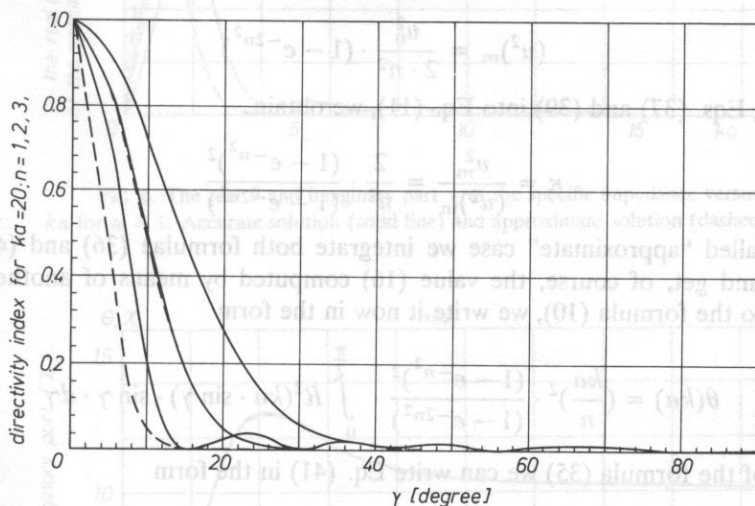

FIG. 5. Directivity index for $ka = 20$ versus γ degrees for $n = 1; 2; 3$.
Accurate solution (solid line) and approximate solution (dashed line).

where $R^\infty(ka \cdot \sin \gamma)$ denotes the approximate value.

The enclosed figures (Fig. 1–5) represent the directivity index versus the angle γ for different values of ka. The continuous line represents the „accurate" values, the dashed line the approximated ones. It is easily seen that the difference between both values is considerable for $n = 1$, still observable, but small for $n = 2$ and practically does not exist for $n = 3$. We notice also that that difference decreases when ka increases. We see, therefore, that for $n = 2$ and $ka > 6$ it is possible to use approximate formulae instead of accurate ones.

4. Specific impedance of the source computed from the accurate formulae

The real part of the specific impedance is computed from the formula (10) substituting for $R(ka \sin \gamma)$ its value (25). In the formula (10) the coefficient κ appears and express itself by the formula (11). We thus begin with the computation of that coefficient. We have in turn: the mean velocity amplitude on the piston

$$u_m = \frac{1}{\Pi a^2} \cdot \int\limits_0^{2\pi} d\varphi \int\limits_0^a u(r) \cdot dr = \frac{2 \cdot u_0}{a^2} \int\limits_0^a e^{-(\frac{n \cdot r}{a})^2} \cdot r \cdot dr \qquad (36)$$

Performing elementary integration, we get

$$u_m = \frac{u_0}{n^2} \cdot (1 - e^{-n^2}) \qquad (37)$$

The mean value of the square of the velocity amplitude equals

$$(u^2)_m = \frac{2 \cdot u_0^2}{a^2} \int\limits_0^a e^{-2(\frac{n \cdot r}{a})^2} \cdot r \cdot dr \qquad (38)$$

The integral (38) is an elementary one, too and we get

$$(u^2)_m = \frac{u_0^2}{2 \cdot n^2} \cdot (1 - e^{-2n^2}) \qquad (39)$$

Substituting Eqs. (37) and (39) into Eq. (11), we obtain

$$\kappa = \frac{u_m^2}{(u^2)_m} = \frac{2}{n^2} \cdot \frac{(1 - e^{-n^2})^2}{(1 - e^{-2n^2})} \qquad (40)$$

In the so-called "approximate" case we integrate both formulae (36) and (40) from 0 to infinity and get, of course, the value (16) computed by means of another method. Returning to the formula (10), we write it now in the form

$$\theta(ka) = (\frac{ka}{n})^2 \cdot \frac{(1 - e^{-n^2})^2}{(1 - e^{-2n^2})} \cdot \int\limits_0^{\frac{\pi}{2}} R^2(ka \cdot \sin \gamma) \cdot \sin \gamma \cdot d\gamma \qquad (41)$$

By means of the formula (35) we can write Eq. (41) in the form

$$\theta(ka) = (\frac{ka}{n})^2 \cdot \frac{1}{1 - e^{-2n^2}} \cdot \int\limits_0^{\frac{\pi}{2}} e^{-\frac{1}{2}(\frac{ka \cdot \sin \gamma}{n})^2} \cdot F^2(n, ka \cdot \sin \gamma) \cdot \sin \gamma \cdot d\gamma \qquad (42)$$

We are here computing the imaginary part of the specific impedance ka by means of the formula (14), i.e., by substituting in Eq. (42) $\cosh \psi$ instead of $\sin \gamma$ and integrating from 0 to infinity

$$\chi(ka = (\frac{ka}{n})^2 \cdot \frac{1}{1 - e^{-2n^2}} \cdot \int\limits_0^\infty e^{-\frac{1}{2}(\frac{ka \cdot \cosh \psi}{n})^2} \cdot F^2(n, ka \cdot \cosh \psi) \cdot \cosh \psi \cdot d\psi \qquad (43)$$

The numerical evaluation of (43) is difficult owing to the infinite limit of the integral, and strongly increasing function $\cosh \psi$. To avoid this difficulty we may apply the following

substitution [5]:

$$\cosh \psi = \frac{1}{\sin \varphi} \qquad (44)$$

Of course we have:

$$\sin \psi \cdot d\psi = -\frac{\cos \varphi}{\sin^2 \varphi} \cdot d\varphi \qquad (45)$$
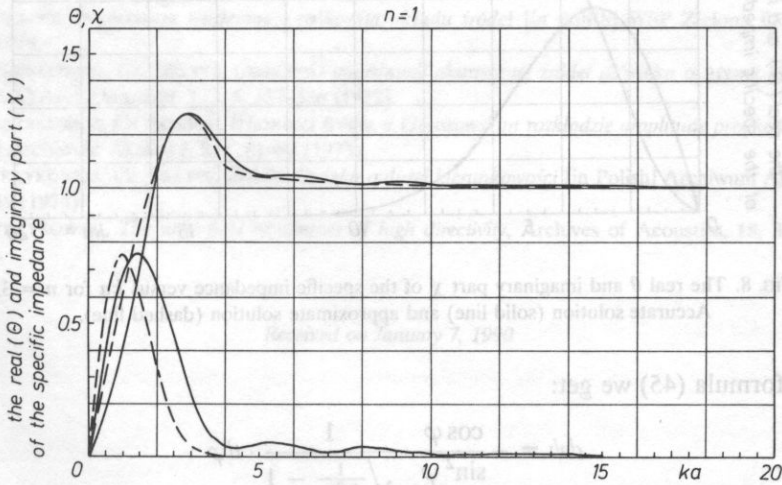

FIG. 6. The real θ and imaginary part χ of the specific impedance versus ka for $n = 1$. Accurate solution (solid line) and approximate solution (dashed line).
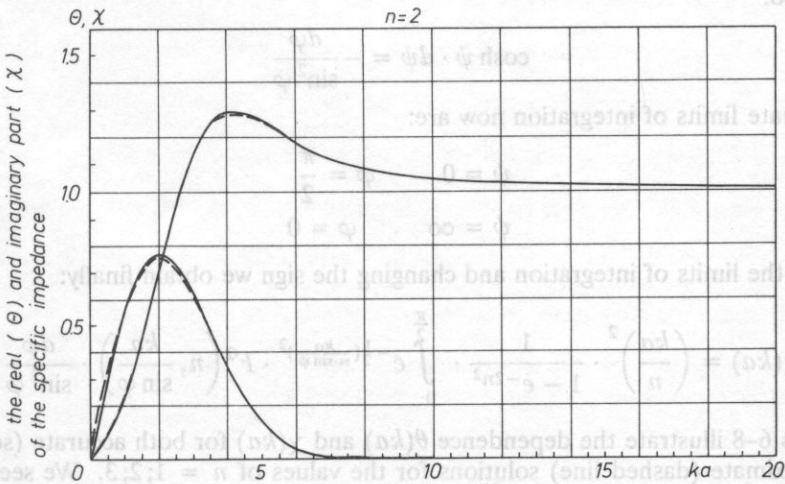

FIG. 7. The real θ and imaginary part χ of the specific impedance versus ka for $n = 2$. Accurate solution (solid line) and approximate solution (dashed line).

FIG. 8. The real θ and imaginary part χ of the specific impedance versus ka for $n = 3$. Accurate solution (solid line) and approximate solution (dashed line).
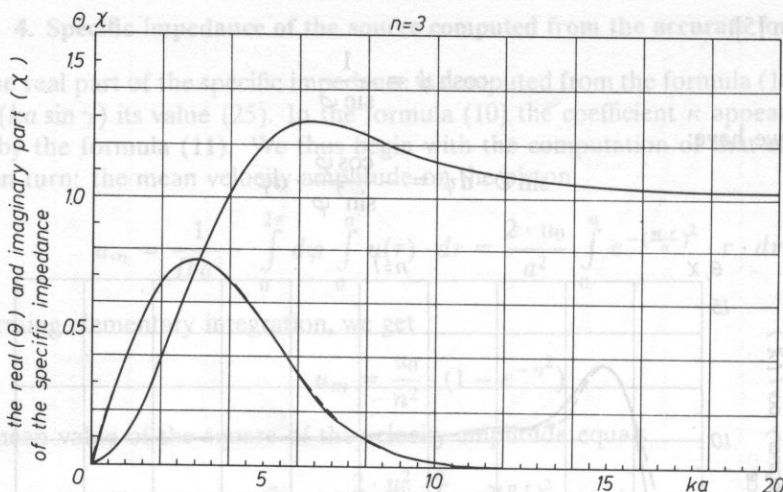
From the formula (45) we get:

$$d\psi = -\frac{\cos\varphi}{\sin^2\varphi} \cdot \frac{1}{\sqrt{\frac{1}{\sin^2\varphi} - 1}} \cdot d\varphi \qquad (46)$$

or:

$$d\psi = -\frac{d\varphi}{\sin\varphi} \qquad (47)$$

We get also:

$$\cosh\psi \cdot d\psi = -\frac{d\varphi}{\sin^2\varphi} \qquad (48)$$

The adequate limits of integration now are:

$$\begin{aligned}\psi &= 0 & \varphi &= \frac{\pi}{2} \\ \psi &= \infty & \varphi &= 0\end{aligned} \qquad (49)$$

Reversing the limits of integration and changing the sign we obtain finally:

$$\chi(ka) = \left(\frac{ka}{n}\right)^2 \cdot \frac{1}{1 - e^{-2n^2}} \cdot \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}(\frac{ka}{n \cdot \sin\psi})^2} \cdot F^2\left(n, \frac{ka}{\sin\varphi}\right) \cdot \frac{d\varphi}{\sin^2\varphi} \qquad (50)$$

Figures 6–8 illustrate the dependence $\theta(ka)$ and $\chi(ka)$ for both accurate (solid line) and approximate (dashed line) solutions for the values of $n = 1; 2; 3$. We see that for $n = 2$ we have practically the same results for approximate and accurate solutions — for $n = 3$ there are no differences. We see also that for $ka > 6$ and $n = 2$ the computed source has very good directivity qualities.

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