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# CALCULATIONS OF EFFECTIVE MATERIAL TENSORS AND THE ELECTROMECHANICAL COUPLING COEFFICIENT OF A TYPE 1-3 COMPOSITE TRANDSUCER

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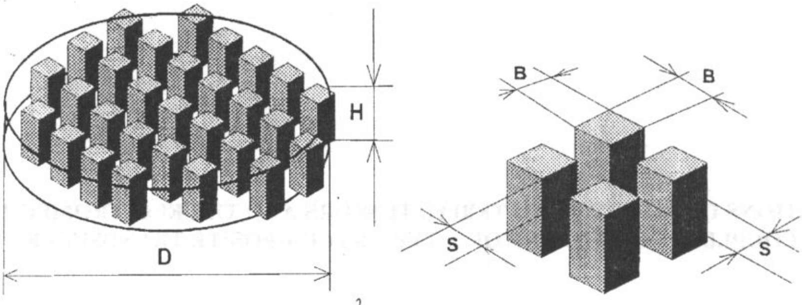
In this paper the purpose and manner of conducting the process of asymptotic homogenization of a type 1-3 composite structure are presented. The formulation of the homogenization process is reduced to numerical static analysis of an elementary symmetry cell of the composite with generalized forces applied at the boundaries of material phases. It is demonstrated that the effective values of the material tensors of the composite depend not only on the tensors of the component materials, but also on variability course of the aforementioned tensors defined over the volume of the solid of an elementary symmetry unit of the composite. The latter factor becomes particularly significant in the case of a step-like discontinuity which occurs, e.g., in the type 1-3 composite structure.

## 1. Introduction

A type 1-3 composite piezoelectric and polymer ultrasound transducer (Fig. 1), used in ultrasonic medical diagnosis, is characterized by a number of desirable properties compared with a typical piezoceramic transducer made from lead titanate and zirconate.

These properties are as follows:

1 – Acoustic impedance of the composite equal to 8-10 MRayl is matched better, acoustically to anatomical tissue with impedance of 1.5 MRayl than piezoceramics with impedance of 33 MRayl.




D = 20 mm  $H = \frac{\lambda}{2}$  B = 200 μm  S = 240 μm

Fig. 1. Structure of the type 1-3 composite transducer.

2 – Reduction of the energy coefficient of the reflection at the transducer-tissue boundary (which results from property 1), from 85% for piezoceramics to 50% for the composite.

3 – Energy efficiency of the transducer, measured by the value of the electromechanical coupling coefficient, is half as high for the composite (60%) than for the piezoceramics (40%).

4 – The greater broad-band width of the composite transducer than that of the piezoceramic one because it is not necessary to use thin quaterwave layers for the reason given in point 1.

5 – The technological workability of forming composite solids with a predetermined curvature radius, eliminating the necessity of using acoustic lenses if it is necessary to obtain focussed heads.

6 – The possibility of making a dynamically focussed head without having to work the material of the transducer itself – by depositing one of the electrodes on the composite transducer, in the form of insulated concentric metallized rings.

The analytical calculation of the electroacoustic quantities of the composite transducer is extremely difficult because all the three dimensions of the smallest symmetry element of the transducer are commensurable with the length of the transmitted longitudinal ultrasonic wave (a threedimensional problem).

An alternative approach is based on the finite element method (FEM) in which the calculations of the electromechanical coupling coefficient consist in the determination of the energy quantities of both the electromagnetic field and the stress and strain fields in the transducer solid.

In practice, the energy quantities can be calculated using the FEM only for a small fragment of the composite soild because of the second power increasing order of magnitude of the rigidity matrix as the total number of degrees of freedom increases. It is possbile to perform, on the other hand, a dynamic analysis of the whole composite solid by dividing the numerical problem into the two following stages:

1 – A static analysis of the elementary symmetry unit of the composite, leading to the determination of substitute material tensors of a hypothetical homogeneous structure.
2 – A dynamic analysis of a substitute homogenous composite structure of the whole solid [2]. In the proposed, modified process of the numerical solution, the FEM mesh nodes is extended twice, and independently of one another. Each time the FEM network generated in this way reaches a degree of densification which is upper-bound only by the order of the global rigidity matrix permitted by the computing applied environment. At the first stage, it applies only to the volume limited to a single elementary cell, and the large density of the FEM network on the elementary volume means high accuracy of calculations of substitute material tensors. At the next homogeneous as a whole, and the homogenized material does not require the necessary densification of the FEM network close to the discontinuity zones – which would be necessary in the composite. As an effect, it becomes possible to perform a dynamic analysis of the whole solid with a much reduced total number of degrees of freedom relative to the unmodified FEM solution. This is achieved without diminishing the calculation accuracy.

## 2. Asymptotic homogenization of the elementary cell of a transducer

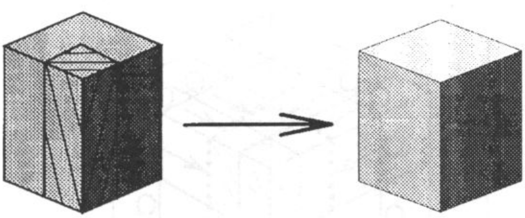


Fig. 2. The idea of homogenization of the elementary cell of the composite transducer.

The purpose of homogenization of the elementary cell of a composite transducer (Fig. 2) is to determine the substitute elementary symmetry unit built of hypothetical homogeneous material. This material shows values of effective material tensors related to:

1 – The averaged values of the tensors of the component materials of the composite

2 – A step-like character of changes in the physical properties at the boundaries of the material phases in the volume of the elementary symmetry unit of the composite solid (3).

The process of homogenization of the composite soild can begin with asymptotic transformation [4]. A composite soild with a periodical structure consisting of the volume $\Omega$ is considered (Fig. 3). The object under consideration is only the solid $\Omega$ with its total size much larger than that of its single elementary symmetry unit $Y$.

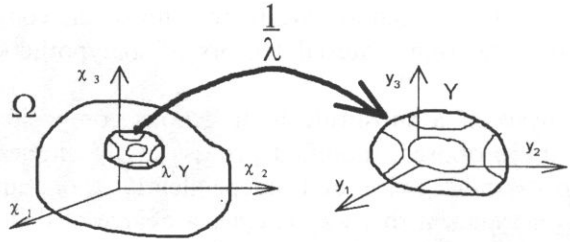



Fig. 3. A transformation of the coordinate system.
$(\chi_1, \chi_2, \chi_3)$ – the global coordinate system,
$(y_1, y_2, y_3)$ – the local coordinate system,
$\Omega$ – the composite solid,
$Y$ – the symmetry element.

The matrix notation will be employed throughout in the text. The tensorial notation will be used at places with specific comments, summation over twice repeated indices being employed. The above assumption is defined in equation (1), where dia ($\Psi$) is the characteristic dimension of the solid ($\Psi$).

$$\lambda = \frac{\mathrm{dia}(Y)}{\mathrm{dia}(\Omega)} \ll 1. \tag{1}$$

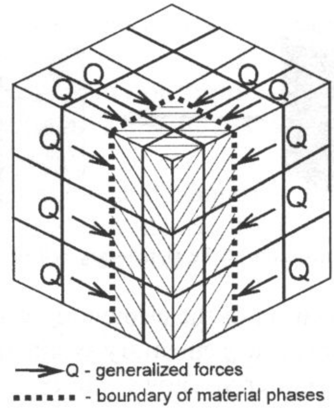



Fig. 4. Static analysis with the division 4x4x3 of the elementary.

The assumption formulated in equation (1) provides the basis for an asymptotic analysis of the solid of composite material with the parameter $\lambda$ tending to zero (Fig. 3) [6].

In keeping with Fig. 4, the coordinate system is transformed from the global system of the solid $(\chi_1, \chi_2, \chi_3)$ to the local one $(y_1, y_2, y_3)$ related to a single symmetry element of the composite. In the course of the transformation, the scale of the axes of the coordinate system changes, as shown by equation (3).

$$x_i = \bar{\mathrm{x}}_i + y_i, \tag{2}$$

$$\{\mathbf{y}_i\} = \frac{\{\mathbf{x}_i\}}{\lambda} \tag{3}$$

In equation (2), $\mathbf{x}_i$ is the tracking vector, while in (3) $\{x_i\}$ and $\{y_i\}$ are the lengths of unit vectors of the axes $x_i$ and $y_i$, respectively.

Considering (2) and (3), the partial derivative with respect to the variable $x_i$ can be written in the form of (4),

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \bar{x}_i} + \frac{1}{\lambda}\frac{\partial}{\partial y_i}. \tag{4}$$

In the further analysis, the symmetrical gradient operator sym $\nabla_{x_i}$(5) and the divergence operator $\mathrm{div}_{x_i}$ (6) will be applied

$$\mathrm{Sym}\,\nabla_{x_i} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}, \tag{5}$$

$$\mathrm{div}_{x_i} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}. \tag{6}$$

The notation of the partial derivative with respect to the variable $x_i$ in (4) determines the form of the Nabel operator (7), the symmetrical gradient operator (8), the gradient operator (9) and the divergence operator (10).

$$\nabla_{x_i} = \nabla_{\bar{x}_i} + \frac{1}{\lambda}\nabla_{y_i}, \tag{7}$$

$$\mathrm{sym}\,\nabla_{x_i} = \mathrm{sym}\,\nabla_{\bar{x}_i} + \frac{1}{\lambda}\,\mathrm{sym}\,\nabla_{y_i}, \tag{8}$$

$$\text{grad}_{x_i} = \text{grad}_{\bar{x}_i} + \frac{1}{\lambda}\text{grad}_{y_i}, \tag{9}$$

$$\text{div}_{x_i} = \text{div}_{\bar{x}_i} + \frac{1}{\lambda}\text{div}_{y_i}, \tag{10}$$

In keeping with (11) and (12), the two independent variables of the electroacoustic field, the displacement $u$ and the potential $\Phi$, can be expanded into convergent asymptotic expansions:

$$u(\bar{x}, y) = u_0(\bar{x}, y) + \lambda u_1(\bar{x}, y) + \lambda^2 u_2(\bar{x}, y) + \ldots, \tag{11}$$

$$\Phi_0(\bar{x}, y) = \Phi_0(\bar{x}, y) + \lambda\Phi_1(\bar{x}, y) + \lambda^2\Phi(\bar{x}, y) + \ldots \tag{12}$$

The problem of the electroacoustic field wave propagation in the composite material containing a piezoelectric component is formulated in the form of a system of six equations (13) – (18):

$$\text{div}_x \mathbf{T}(\bar{x}, y) = \rho(y)\ddot{\mathbf{u}}(\bar{x}, y), \tag{13}$$

$$\text{div}_x \mathbf{D}(\bar{x}, y) = 0, \tag{14}$$

$$\mathbf{T}(\bar{x}, y) = c(y)\mathbf{S}(\bar{x}, y) - e(y)\mathbf{E}(\bar{x}, y), \tag{15}$$

$$\mathbf{D}(\bar{x}, y) = e(y)\mathbf{S}(\bar{x}, y) + \varepsilon(y)\mathbf{E}(\bar{x}, y), \tag{16}$$

$$\mathbf{S}(\bar{x}, y) = \text{sym}_x u(\bar{x}, y), \tag{17}$$

$$\mathbf{E}(\bar{x}, y) = -\nabla_x \Phi(\bar{x}, y), \tag{18}$$

where **u** is the vector of mechanical displacement, $\Phi$ is the scalar of the electric potential, **S** is the displacement tensor, **T** is the stress tensor, **E** is the vector of the intensity of the electric field, **D** is the vector of electric induction, $\rho$ is the scalar of mass density, **c** is the tensor of mechanical rigidity, **e** is the piezoelectric tensor and $\varepsilon$ is the dielectric tensor.

Thus, equation (13) is a notation of Newton's second principle of dynamics for a continuous medium. Equation (14) is one of four Maxwell equations, stating that in the volume of the composite solid there are no free electric charges. Dependencies (15) and (16) contain a notation of the simple and converse piezoelectric effects which take place in the composite solid. Through (15) and (16), in the piezoelectric material there occurs the effect of coupling of the stresses and strains field with the electromagnetic one (through the nonzero tensor **e**), which thus form one electroacoustic field. Dependence (17) is a principle which is generally valid in the continuous medium mechanics, linking the mechanical quantities of strain and displacement. Equation (18) determines the relation between the quantities of the electric field the intensity and the potential. The expansion of the two independent variables of the electroacoustic field, the mechanical displacement $u$ and the electric potential $\Phi$, into convergent asymptotic series with respect to $\lambda$ implies asymptotic expansions of all the derivative quantities (19) – (22):

$$\mathbf{S}(\bar{x}, y)= \lambda^{-2}S_{-2}(\bar{x},y)+\lambda^{-1}S_{-1}(\bar{x},y)+S_{0}(\bar{x},y)+\lambda S_{1}(\bar{x},y)+\lambda^{2}S_{2}(\bar{x},y)+ \dots \quad (19)$$

$$\mathbf{T}(\bar{x}, y) = \lambda^{-2}T_{-2}(\bar{x},y)+\lambda^{-1}T_{-1}(\bar{x},y)+T_{0}(\bar{x},y)+\lambda T_{1}(\bar{x},y)+\lambda^{2}T_{2}(\bar{x},y)+ \dots \quad (20)$$

$$\mathbf{E}(\bar{x}, y) = \lambda^{-2}E_{-2}(\bar{x},y)+\lambda^{-1}E_{-1}(\bar{x},y)+E_{0}(\bar{x},y)+\lambda E_{1}(\bar{x},y)+\lambda^{2}E_{2}(\bar{x},y)+ \dots \quad (21)$$

$$\mathbf{D}(\bar{x}, y)= \lambda^{-2}D_{-2}(\bar{x},y)+\lambda^{-1}D_{-1}(\bar{x},y)+D_{0}(\bar{x},y)+\lambda D_{1}(\bar{x},y)+\lambda^{2}D_{2}(\bar{x},y)+ \dots \quad (22)$$

The asymptotic expansions of equations (19)–(22) will also contain terms involving the powers of $\lambda$ equal to minus one and minus two. It results from the fact that, in keeping with (17, 18 and 15, 16), all the derivatives (19)–(22) are functions od the mechanical displacement $u$ and the electric potential $\Phi$ acted by the differential operators (7)–(10). In turn, operators (7)–(10) contain the partial derivatives (5, 6) in their structure. On the other hand, the partial derivatives (4) contain a component including the fraction $1/\lambda$. It is exactly the component that causes decreasing order of $\lambda^{k}$ by one.

Subsequently the values of the **u**, $\Phi$, **S**, **T**, **E**, **D** in equations (13) – (18) can be replaced by their infinite power expansions (11) – (12), (19) – (22). Then on both sides of the system of equations (13-18), there are only infinite power series with the given small parameter $\lambda$. The terms on both sides of the equations related to the same powers of $\lambda$ are equal to one another. The terms containing the zero power of $\lambda$ (23) – (28) can be compared:

$$S_{0}(\bar{x}, y) = \operatorname{sym}\nabla_{\bar{x}}u_{0}(\bar{x}, y), \quad (23)$$

$$E_{0}(\bar{x}, y) = -\nabla_{\bar{x}}\Phi(\bar{x}, y)- \nabla_{y}\Phi(\bar{x}, y), \quad (24)$$

$$T_{0}(\bar{x}, y) = c(y)S_{0}(\bar{x}, y)-e(y)E_{0}(\bar{x}, y) \quad (25)$$

$$D_{0}(\bar{x}, y) = e(y)S_{0}(\bar{x}, y) - \varepsilon(y)E_{0}(\bar{x}, y) \quad (26)$$

$$\operatorname{div}_{y}T_{1}(\bar{x}, y) +\operatorname{div}_{\bar{x}}T_{0}(\bar{x}, y) = \rho(y)\ddot{u}x(\bar{x}, y), \quad (27)$$

$$\operatorname{div}_{y}D_{1}(\bar{x}, y)+\operatorname{div}_{\bar{x}}D_{0}(\bar{x}, y) = 0. \quad (28)$$

Then, the same can be done for the terms which contain the power $\lambda$ equal to minus one (29) – (34):

$$S_{-1}(\bar{x}, y) = \operatorname{sym}\nabla_{y}u_{0}(\bar{x}, y), \quad (29)$$

$$E_{-1}(\bar{x}, y) = -\nabla_{y}\Phi_{0}(\bar{x}, y), \quad (30)$$

$$T_{-1}(\bar{x}, y) = c(y)S_{-1}(\bar{x}, y) - e(y)E_{-1}(\bar{x}, y), \quad (31)$$

$$D_{-1}(\bar{x}, y) = e(y)S_{-1}(\bar{x}, y)-\varepsilon(y)E_{-1}(\bar{x}, y), \quad (32)$$

$$\mathrm{div}_y T_0(\bar{x}, y) + \mathrm{div}_{\bar{x}} T_{-1}(\bar{x}, y) = 0\,, \tag{33}$$

$$\mathrm{div}_y D_0(\bar{x}, y) + \mathrm{div}_{\bar{x}} D_{-1}(\bar{x}, y) = 0\,, \tag{34}$$

Finally, the terms containing the power of $\lambda$ equal to minus two (35)–(40) can be compared:

$$S_{-2}(\bar{x}, y) = 0\,, \tag{35}$$

$$E_{-2}(\bar{x}, y) = 0\,, \tag{36}$$

$$T_{-2}(\bar{x}, y) = 0\,, \tag{37}$$

$$D_{-2}(\bar{x}, y) = 0\,, \tag{38}$$

$$\mathrm{div}_y T_{-1}(\bar{x}, y) = 0\,, \tag{39}$$

$$\mathrm{div}_y D_{-1}(\bar{x}, y) = 0\,. \tag{40}$$

The following two important dependencies result from equations (39)–(40) and (29)–(30):

$$\mathrm{div}_y \Big[ c(y)\, \mathrm{sym}\, \nabla_y u_0(\bar{x}, y) - e(y) \nabla_y \Phi_0(\bar{x}, y) \Big] = 0\,, \tag{41}$$

$$\mathrm{div}_y \Big[ e(y)\, \mathrm{sym}\, \nabla_y u_0(\bar{x}, y) - \varepsilon(y) \nabla_y \Phi_0(\bar{x}, y) \Big] = 0\,, \tag{42}$$

The system of equations (41)–(42) is satisfied in terms of identity only when the dependencies specified in equations (43 – 44) occur:

$$u_0(\bar{x}, y) = u_0(\bar{x})\,, \tag{43}$$

$$\Phi_0(\bar{x}, y) = \Phi_0(\bar{x})\,. \tag{44}$$

From (43)–(44), an important conclusion results concerning the terms of the zero order of the expansion of the two independent variables of the electroacoustic field, the displacement **u** (11) and the potential $\Phi$ (12), into power series. On the basis (43) – (44), it can be stated that these terms are independent of the local variable $y$.

Considering dependencies (43)–(44) in the system of equations (29)–(34), the following system of equations is obtained (45)–(50):

$$S_{-1}(\bar{x}, y) = 0\,, \tag{45}$$

$$E_{-1}(\bar{x}, y) = 0\,, \tag{46}$$

$$T_{-1}(\bar{x}, y) = 0\,, \tag{47}$$

$$D_{-1}(\bar{x}, y) = 0\,, \tag{48}$$

$$\mathrm{div}_y T_0(\bar{x}, y) = 0\,, \tag{49}$$

$$\mathrm{div}_y D_0(\bar{x}, y) = 0\,. \tag{50}$$

For a value of $\lambda$ much lower than unity (for a solid with the geometry shown in Fig. 1, $\lambda$ is approximately equal to 0.011), it is permissible to neglect in the asymptotic expansion of any electroacoustic quantity $\Psi$ all terms, apart from $\lambda$, related to the zero power (1). This means the approximation of the exact value $\Psi$ of the term $\Psi_0$ of its asymptotic expansion. Then, the effecxtive value $\Psi_E(x)$ of any electroacoustic quantity $\Psi$ on the volume of the symmetry element $Y$ of the asymptotic expansion $\Psi$(45). The volume of the symmetry element $Y$ is designated as vol($Y$).

$$\Psi_E(\overline{x}) \equiv \| \Psi_0(\overline{x}, y) \|_y \equiv \frac{1}{\mathrm{vol}(Y)} \oiiint_Y \Psi_0(\overline{x}, y) dY. \tag{51}$$

As a result of applying operator (51), on both sides, in the system of equations (13) −(18), the system of equations (52)−(57), describing the relations between the effective quantities, that is, those averaged over the volume of the elementary cell $Y$, according to formula (45). The sought quantities are $c^{EF}$, $e^{EF}$, $\varepsilon^{EF}$ – the effective values of the material tensors of the substitute homogeneous piezoelectric material with respect to the two-component composite in question.

$$\mathrm{div}_x T_E(\overline{x}) = \rho^{EF} \ddot{u}_E(\overline{x}), \tag{52}$$

$$\mathrm{div}_x D_E(\overline{x}) = 0, \tag{53}$$

$$T_E(\overline{x}) = c^{EF} S_E(\overline{x}) - e^{EF} E_E(\overline{x}), \tag{54}$$

$$D_E(\overline{x}) = e^{EF} S_E(\overline{x}) + \varepsilon^{EF} E_E(\overline{x}), \tag{55}$$

$$S_E(\overline{x}) = \mathrm{sym} \nabla_x u_E(\overline{x}), \tag{56}$$

$$E_E(\overline{x}) = -\nabla_x \Phi_E(\overline{x}). \tag{57}$$

From equations (23) – (24), (43) – (44) and (56) – (57), the following dependencies (58) – (59) result:

$$S_0(\overline{x}, y) = S_E(\overline{x}) + \mathrm{sym} \nabla_y u_1(\overline{x}, y), \tag{58}$$

$$E_0(\overline{x}, y) = E_E(\overline{x}) - \nabla_y \Phi_1(\overline{x}, y), \tag{59}$$

From equations ((49) – (50), (23) – (26) and (58) – (59)), the following dependencies (60) – (61) result:

$$\mathrm{div}_y \left[ c(y) \cdot \mathrm{sym} \nabla_y u_1(\overline{x}, y) - e(y) \cdot \nabla_y \Phi_1(\overline{x}, y) \right] = -\mathrm{div}_y c(y) \cdot S_E(\overline{x}) + \\ + \mathrm{div}_y e(y) \cdot E_E(\overline{x}), \tag{60}$$

$$\mathrm{div}_y \left[ e(y) \cdot \mathrm{sym} \nabla_y u_1(\overline{x}, y) + \varepsilon(y) \cdot \nabla_y \Phi_1(\overline{x}, y) \right] = -\mathrm{div}_y \varepsilon(y) \cdot S_E(\overline{x}) + \\ + \mathrm{div}_y \varepsilon(y) \cdot E_E(\overline{x}). \tag{61}$$

In keeping with equations (62), (63), the following auxiliary tensors **A(y), B(y), G(y), H(y),** which mutually link $u_1(x, y)$, $\Phi_1(x, y)$ with $S_E(\mathrm{x})$, $E_E(x)$, are then derived:

$$u_1(\overline{x}, y) = \mathbf{A(y)} \cdot S_E(\overline{x}) + \mathbf{B(y)} \cdot E_E(\overline{x}), \tag{62}$$

$$\Phi_1(\overline{x}, y) = \mathbf{G(y)} \cdot S_E(\overline{x}) + \mathbf{H(y)} \cdot E_E(\overline{x}). \tag{63}$$

From equations (60), (61), (62), (63), the system of equations (64) – (67) is obtained, the solution of which are the values of the auxiliary quantities of the tensors **A(y), B(y), G(y), H(y)**:

$$\mathrm{div}_y\left[c(y)\cdot \mathrm{sym}\,\nabla_y \mathbf{A}(\mathbf{y}) - e(y)\cdot \nabla_y \mathbf{G}(\mathbf{y})\right] = -\mathrm{div}_y c(y), \tag{64}$$

$$\mathrm{div}_y\left[e(y)\cdot \mathrm{sym}\,\nabla_y \mathbf{A}(\mathbf{y}) + \varepsilon(y)\cdot \nabla_y \mathbf{G}(\mathbf{y})\right] = -\mathrm{div}_y e(y), \tag{65}$$

$$\mathrm{div}_y\left[c(y)\cdot \mathrm{sym}\,\nabla_y \mathbf{B}(\mathbf{y}) - e(y)\cdot \nabla_y \mathbf{H}(\mathbf{y})\right] = -\mathrm{div}_y e(y), \tag{66}$$

$$\mathrm{div}_y\left[e(y)\cdot \mathrm{sym}\,\nabla_y \mathbf{B}(\mathbf{y}) + \varepsilon(y)\cdot \nabla_y \mathbf{H}(\mathbf{y})\right] = -\mathrm{div}_y \varepsilon(y). \tag{67}$$

The system of equations (64)–(67) can be expressed using the tensor notation (68)–(71):

$$\left[c_{ijkl}\,\mathbf{A}^{pq}_{k,l} - e_{kij}\,\mathbf{G}^{pq}_{,k}\right]_{,j} = -c_{ijpq,j}, \tag{68}$$

$$\left[e_{ikl}\,\mathbf{A}^{pq}_{k,l} + \varepsilon_{ik}\,\mathbf{G}^{pq}_{,k}\right]_{,i} = -e_{ipq,i}, \tag{69}$$

$$\left[c_{ijkl}\,\mathbf{B}^{p}_{k,l} - e_{kij}\,\mathbf{H}^{p}_{,k}\right]_{,j} = e_{pij,j}, \tag{70}$$

$$\left[e_{ikl}\,\mathbf{B}^{p}_{k,l} + \varepsilon_{ik}\mathbf{H}^{pq}_{,k}\right]_{,i} = -e_{ip,i}. \tag{71}$$

Equations (64)–(67) and (68)–(71) are static equations (Fig. 3) with generalized excitations in the form of generalized forces (72)–(73) and generalized charges (74)–(75) as well as generalized inputs in the form of generalized forces (76)–(77) and generalized charges (78)–(79) shown below.

The generalized excitations in terms of generalized forces (72)–(73) are:

$$\overline{\mathbf{T}}^{kl}_{ij} = c_{ijpq}\mathbf{A}^{kl}_{p,q} - e_{pij}\mathbf{G}^{kl}_{,p}, \tag{72}$$

$$\overline{\mathbf{T}}^{k}_{ij} = c_{ijpq}\,\mathbf{B}^{k}_{p,q} - e_{pij}\mathbf{H}^{k}_{,p}. \tag{73}$$

The generalized response expressed in context of generalized charges (74)–(75) are:

$$\overline{\mathbf{D}}^{ij}_{k} = e_{kpq}\,\mathbf{A}^{ij}_{p,q} + \varepsilon_{kp}\mathbf{G}^{ij}_{,p}, \tag{74}$$

$$\overline{\mathbf{D}}^{j}_{i} = e_{ipq}\,\mathbf{B}^{j}_{p,q} + \varepsilon_{ip}\,\mathbf{H}^{j}_{,p}. \tag{75}$$

The generalized inputs in the form of generalized forces (76)–(77) are as follows: where $\mathbf{n}_j = \{n_1, n_2, n_3\}$ is the vector of the external normal vector.

$$(\mathbf{F})_i^{pq} \equiv \left[\ |c_{ijpq}|\ \right] \cdot \mathbf{n}_j, \tag{76}$$

$$(\mathbf{F})_i^{p} \equiv \left[\ |e_{pij}\ |\ \right] \cdot \mathbf{n}_j, \tag{77}$$

The generalized inputs in the form of generalized charges (78)–(79) are:

$$(\mathbf{Q})^{pq} \equiv \left[\ |e_{ipq}\ |\ \right] \cdot \mathbf{n}_i, \tag{78}$$

$$(\mathbf{Q})^{pq} \equiv \left[\ |\varepsilon_{ipq}\ |\ \right] \cdot \mathbf{n}_i. \tag{79}$$

The static problem formulated in the form of the equation of statics (68)–(71) is illustrated in Fig. 4. For the system of equations (68)–(71), predetermined loads on the boundaries of material phases on the volume of the symmetry element $Y$ of the composite solid were determined in the form of generalized forces (76)–(77) and generalized charges (78)–(79) which are induced by a step-like change in the values of the material tensors on the boundaries of the phases in the volume of the aformentioned element $Y$ (Fig. 3). The boundary conditions mentioned above are reflected in the form of the right-hand side of the system of equations (64)–(67) and (68)–(71).

The mechanical rigidity matrices for piezoceramics and polymers have the form:

PIEZOCERAMICS

$$\begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ . & c_{1111} & c_{1133} & 0 & 0 & 0 \\ . & . & c_{3333} & 0 & 0 & 0 \\ . & . & . & c_{2323} & 0 & 0 \\ . & \text{symmetry} & . & . & c_{2323} & 0 \\ . & . & . & . & . & c_{1212} \end{bmatrix}$$

POLYMERS

$$\begin{bmatrix} c_{1111} & c_{1122} & c_{1122} & 0 & 0 & 0 \\ . & c_{1111} & c_{1122} & 0 & 0 & 0 \\ . & . & c_{2222} & 0 & 0 & 0 \\ . & . & . & c_{2222} & 0 & 0 \\ . & \text{symmetry} & . & . & c_{2222} & 0 \\ . & . & . & . & . & c_{1212} \end{bmatrix}$$

The dielectric rigidity matrices for piezoceramics and polymers:

PIEZOCERAMICS

$$\begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ . & \varepsilon_{11} & 0 \\ \text{symmetry} & . & \varepsilon_{33} \end{bmatrix}$$

POLYMERS

$$\begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ . & \varepsilon_{11} & 0 \\ \text{symmetry} & . & \varepsilon_{11} \end{bmatrix}.$$

The piezoelectric rigidity matrices for piezoceramics and polymers:

PIEZOCERAMICS

$$\begin{bmatrix} 0 & 0 & e_{311} \\ 0 & 0 & e_{311} \\ 0 & 0 & e_{333} \\ 0 & e_{223} & 0 \\ e_{223} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In view of the form of the material tensors $c_{ijkl}$, $e_{kij}$, $\varepsilon_{ik}$ for polarized piezoceramics (orthotropic material) and polymers (isotropic properties), most terms in the matrix representation of the aformentioned tensors beyond the main diagonal are zero in value. Equations (80)–(82) show the proposed form of the formula for the effective calculation of the values of the material tensors $c^{EF}$, $e^{EF}$, $\varepsilon^{EF}$ for the substitute homogeneous piezoelectric material.

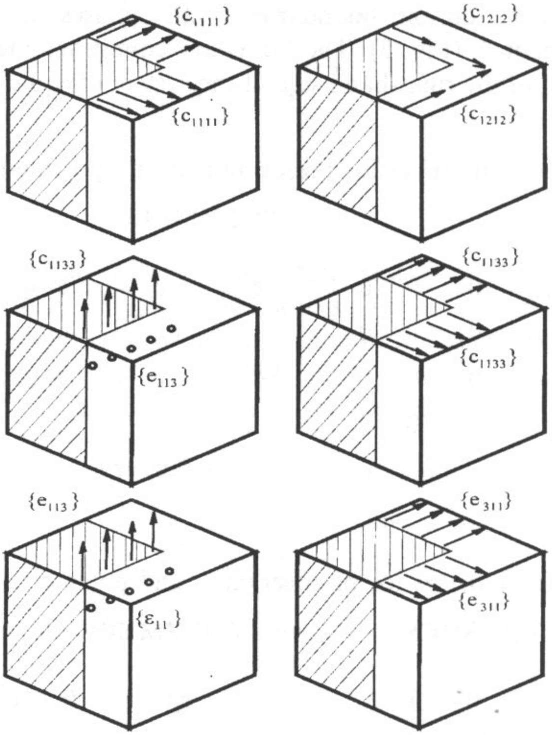



Fig. 5. Six elementary static related to the load of the elementary symmetry unit of the composite caused by a step-like change in the value of the material tensors at the boundaries of material phases.

$$c_{ijkl}^{EF} = \|c_{ijkl}\|_Y + \|\overline{\mathbf{T}}_{ij}^{kl}\|_Y, \tag{80}$$

$$\varepsilon_{ij}^{EF} = \|\varepsilon_{ij}\|_Y + \|\overline{\mathbf{D}}_i^{j}\|_Y, \tag{81}$$

$$e_{ijk}^{EF} = \|e_{ijk}\|_Y + \|\overline{\mathbf{T}}_{ij}^{p}\|_Y = - \|e_{ijk}\|_Y + \|\overline{\mathbf{D}}_k^{ij}\|_Y. \tag{82}$$

In equations (80)–(82), two terms were separated on the right-hand side of each equation. The first is related to the weighted average of the value of the material tensors of the composite components on the volume of the solid of the elementary cell. The other illustrates, in turn, the impact of the step-like discontinuity of the value of the aforementioned tensors on the volume of the solid in question.

In keeping with equations (76)–(79), the static analysis shown in Fig. 4 is reduced to the solution of six independent static problems with load cases presented in Fig. 5.

On the basis of (80)–(82), calculations of the values of the components of the tensors were made and they are listed in Table 1.

**Table 1.** Calculations of the values of the components of the material tensors of piezoceramics, an epoxy resin and the composite made from two afromentioned materials.

| | | Piezoceramics | Epoxy resin | Composite subject homogenization |
|---|---|---|---|---|
| $c_{11}$ | kg/(m*s$^2$) | 12.1*10$^{10}$ | 7.4*10$^{10}$ | 3.09*10$^{10}$ |
| $c_{12}$ | kg/(m*s$^2$) | 7.5*10$^{10}$ | 1.4*10$^{10}$ | 1.65*10$^{10}$ |
| $c_{13}$ | kg/(m*s$^2$) | 7.7*10$^{10}$ | 1.4*10$^{10}$ | 1.63*10$^{10}$ |
| $c_{33}$ | kg/(m*s$^2$) | 7.3*10$^{10}$ | 7.4*10$^{10}$ | 2.91*10$^{10}$ |
| $c_{44}$ | kg/(m*s$^2$) | 2.1*10$^{10}$ | 4.7*10$^{10}$ | 9.09*10$^{10}$ |
| $c_{66}$ | kg/(m*s$^2$) | 2.3*10$^{10}$ | 4.7*10$^{10}$ | 8.49*10$^{10}$ |
| $\varepsilon_{11}$ | A$^{2}$*s$^4$/kg*m$^3$ | 14.6*10$^{-9}$ | 3.53*10$^{-11}$ | 1.08*10$^{-9}$ |
| $\varepsilon_{33}$ | A$^{2}$*s$^4$/kg*m$^3$ | 15*10$^{-9}$ | 3.53*10$^{-11}$ | 1.48*10$^{-9}$ |
| $e_{31}$ | A*s/m$^2$ | -5.4 | 0 | -1.66 |
| $e_{33}$ | A*s/m$^2$ | 15.1 | 0 | 3.19 |
| $e_{15}$ | A*s/m$^2$ | 12.3 | 0 | 2.60 |

## 3. Dynamic analysis of the electromechanical coupling coefficient

The values of the electromechanical coupling coefficient $k$ are calculated using a numerical dynamic analysis involving direct integration within the framework of the finite element method:

$$k = \frac{E_m}{\sqrt{E_{st} E_d}} \tag{83}$$

$$E_m = \frac{1}{4}\,(u^t K_{u\Phi} \Phi + \Phi_t K^{t}_{u\Phi} u), \tag{84}$$

$$E_{st} = \frac{1}{2}\, u^t K_{uu} u, \tag{85}$$

$$E_d = \frac{1}{2}\, \Phi^t K_{\Phi\Phi} \Phi, \tag{86}$$

where $E_m$ is the electromechanical energy, $E_{st}$ is the energy of the field of strains and stresses, $E_d$ is the electric field energy, $u$ is the displacement matrix, $\Phi$ is the potential matrix, $K_{uu}$ is the mechanical matrix rigidity, $K_{\Phi\Phi}$ is the dielectric rigidity matrix and $K^t$ is the matrix transposed with respect to the matrix $K$.

The values of the electromechanical coupling coefficient $k$ were calculated from formula (98) for the composite transducer (Fig. 1) and the PZT transcuder with the same electrical resonance frequency and disk diameter. The results are listed in Table 2.

**Table 2.** Values of the electromechanical coupling coefficient $k$ for the piezoceramic and composite transducers previously subjected to homogenization.

| Transducer type | Electrical resonance frequency MHz | Electromechanical coupling coefficient $k$ |
|---|---|---|
| Piezoceramics | 1.80 | 0.48 |
| Composite previously subjected to homogenization | 1.86 | 0.69 |

## 4. Conclusions

The approximation of the values of the effective material tensors of hypothetical homogeneous material only by weighted averages of its components involves the error related to neglecting the impact of a step-like change (in the case of the type 1-3 composite) in the physical properties at the boundaries of the material phases in the volume of the elementary symmetry unit of the composite. The division of the numerical calculations into two independent stages (static and dynamic analyses) makes it possible to carry them out for the whole composite solid, since in this way the limitation imposed by the permissible order of the global rigidity matrix of the problem is bypassed. The calculations confirmed an approximately 50% increase in the value of the electromechanical coupling coefficient of the composite with respect to the piezoceramics.

## References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